

# OPTIMAL DECAY ESTIMATES IN THE CRITICAL $L^p$ FRAMEWORK FOR FLOWS OF COMPRESSIBLE VISCOUS AND HEAT-CONDUCTIVE GASES

RAPHAËL DANCHIN AND JIANG XU

**ABSTRACT.** The global existence issue in critical regularity spaces for the full Navier-Stokes equations satisfied by compressible viscous and heat-conductive gases has been first addressed in [10], then recently extended to the general  $L^p$  framework in [13]. In the present work, we establish decay estimates for the global solutions constructed in [13], under an additional mild integrability assumption that is satisfied if the low frequencies of the initial data are in  $L^r(\mathbb{R}^d)$  with  $\frac{p}{2} \leq r \leq \min\{2, \frac{d}{2}\}$ . As a by-product in the case  $d = 3$ , we recover the classical decay rate  $t^{-\frac{3}{4}}$  for  $t \rightarrow +\infty$  that has been observed by A. Matsumura and T. Nishida in [30] for solutions with high Sobolev regularity. Compared to a recent paper of us [14] dedicated to the barotropic case, not only we are able to treat the full system, but we also weaken the low frequency assumption and improve the decay exponents for the high frequencies of the solution.

## 1. INTRODUCTION

The motion of general viscous and heat conductive gases is governed by

$$(1.1) \quad \begin{cases} \partial_t \rho + \operatorname{div}_x(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}_x(\rho u \otimes u) + \nabla_x P = \operatorname{div}_x \tau, \\ \partial_t \left[ \rho \left( \frac{|u|^2}{2} + e \right) \right] + \operatorname{div}_x \left[ u \left( \rho \left( \frac{|u|^2}{2} + e \right) + P \right) \right] = \operatorname{div}_x(\tau \cdot u - q), \end{cases}$$

where  $\rho = \rho(t, x) \in \mathbb{R}_+$  denotes the density,  $u = u(t, x) \in \mathbb{R}^d$ , the velocity field and  $e = e(t, x) \in \mathbb{R}_+$ , the internal energy per unit mass. We restrict ourselves to the case of a Newtonian fluid: the viscous stress tensor is  $\tau = \lambda \operatorname{div}_x u \operatorname{Id} + 2\mu D_x(u)$ , where  $D_x(u) \triangleq \frac{1}{2}(D_x u + {}^T D_x u)$  stands for the deformation tensor, and  $\operatorname{div}_x$  is the divergence operator with respect to the spatial variable. The bulk and shear viscosities are supposed to satisfy

$$(1.2) \quad \mu > 0 \quad \text{and} \quad \nu \triangleq \lambda + 2\mu > 0.$$

The heat flux  $q$  is given by  $q = -\kappa \nabla_x \mathcal{T}$  where  $\kappa > 0$ , and  $\mathcal{T}$  stands for the temperature. For simplicity, the coefficients  $\lambda$ ,  $\mu$  and  $\kappa$  are taken constant in all that follows.

It is well known that combining the second and third equations of (1.1) yields

$$\partial_t(\rho e) + \operatorname{div}_x(\rho u e) + P \operatorname{div}_x u - \kappa \Delta_x \mathcal{T} = 2\mu D_x(u) : D_x(u) + \lambda(\operatorname{div}_x u)^2.$$

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In order to reformulate System (1.1) in terms of  $\rho$ ,  $u$  and  $\mathcal{T}$  only, we make the additional assumption that the internal energy  $e = e(\rho, \mathcal{T})$  satisfies Joule law :

$$(1.3) \quad \partial_{\mathcal{T}} e = C_v \text{ for some positive constant } C_v$$

and that the pressure function  $P = P(\rho, \mathcal{T})$  is of the form

$$(1.4) \quad P(\rho, \mathcal{T}) = \pi_0(\rho) + \mathcal{T}\pi_1(\rho),$$

where  $\pi_0$  and  $\pi_1$  are given smooth functions<sup>1</sup>. Then taking advantage of the Gibbs relations for the internal energy and the Helmholtz free energy, we get the Maxwell relation

$$\rho^2 \partial_{\rho} e(\rho, \mathcal{T}) = P(\rho, \mathcal{T}) - \mathcal{T} \partial_{\mathcal{T}} P(\rho, \mathcal{T}) = \pi_0(\rho),$$

and end up with the following temperature equation:

$$(1.5) \quad \rho C_v (\partial_t \mathcal{T} + u \cdot \nabla_x \mathcal{T}) + \mathcal{T} \pi_1(\rho) \operatorname{div}_x u - \kappa \Delta_x \mathcal{T} = 2\mu D_x(u) : D_x(u) + \lambda (\operatorname{div}_x u)^2.$$

We are concerned with the large time behaviour of (strong) global solutions to (1.1) in the case where the fluid domain is the whole space  $\mathbb{R}^d$  with  $d \geq 3$ . We focus on solutions that are close to some constant equilibrium  $(\bar{\rho}, 0, \bar{\mathcal{T}})$  with  $\bar{\rho} > 0$  and  $\bar{\mathcal{T}} > 0$  satisfying the linear stability condition:

$$(1.6) \quad \partial_{\rho} P(\bar{\rho}, \bar{\mathcal{T}}) > 0 \quad \text{and} \quad \partial_{\mathcal{T}} P(\bar{\rho}, \bar{\mathcal{T}}) > 0.$$

Recall that, starting with the pioneering work by Matsumura and Nishida [30], a number of papers have been dedicated to that issue in the case of solutions with high Sobolev regularity. We here aim at performing the long time asymptotics within the so-called *critical regularity* framework, that is in functional spaces endowed with norms that are invariant for all  $\ell > 0$  by the transform:

$$(1.7) \quad \rho(t, x) \rightsquigarrow \rho(\ell^2 t, \ell x), \quad u(t, x) \rightsquigarrow \ell u(\ell^2 t, \ell x), \quad \mathcal{T}(t, x) \rightsquigarrow \ell^2 \mathcal{T}(\ell^2 t, \ell x) \quad \ell > 0.$$

That definition of criticality corresponds to the scaling invariance (up to a suitable change of the pressure terms) of System (1.1) written in terms of  $(\rho, u, \mathcal{T})$ .

Scaling invariance plays a fundamental role in the study of evolutionary PDEs. Recall that in the context of the incompressible Navier-Stokes equations, working in critical spaces goes back to the work by Fujita & Kato in [16] (see also more recent results in Kozono & Yamazaki [25] and Cannone [3]), and that it has been extended to the compressible Navier-Stokes equations in e.g. [4, 7, 6, 8, 10, 11, 13, 17].

Even though, rigorously speaking, System (1.1) does not possess any scaling invariance, it is possible to solve it locally in time in Banach spaces endowed with norms having the invariance given by (1.1). For example, it has been proved in [4, 11] that in dimension  $d \geq 3$ , it is well-posed in  $\dot{B}_{p,1}^{\frac{d}{p}} \times (\dot{B}_{p,1}^{\frac{d}{p}-1})^d \times \dot{B}_{p,1}^{\frac{d}{p}-2}$  if  $1 \leq p < d$ . A key fact for proving that result is to observe that the coupling between the equations is low order.

As regards the global well-posedness issue however, the low order terms have to be taken into account in the choice of a suitable functional framework, and it is suitable to use “hybrid” Besov norms with different regularity indices in the low and high frequencies. The basic heuristics is that, for low frequencies, the first order terms predominate so that (1.1) can be handled by means of hyperbolic methods (in particular it is natural to work at the same level of regularity for the density, velocity and temperature). In contrast, in the high frequency regime, two types of behaviours coexist: the parabolic one for the velocity and temperature, and the damped one for the density. This heuristics may be translated in terms of a priori estimates by means of an energy method that can be directly implemented

<sup>1</sup>One can thus consider e.g. perfect gases ( $\pi_0(\rho) = 0$  and  $\pi_1(\rho) = R\rho$  with  $R > 0$ ) or Van-der-Waals fluids ( $\pi_0(\rho) = -\alpha\rho^2$ ,  $\pi_1(\rho) = \beta\rho/(\delta - \rho)$  with  $\alpha, \beta, \delta > 0$ ).

on (1.1), after spectral localization (the main difficulty arising from the convection term in the density equation can be by-passed thanks to suitable integration by parts and commutator estimates, see [10]). Recently, the first author and L. He [13] extended the results of [10] to more general  $L^p$  Besov spaces, and got the following statement:

**Theorem 1.1.** *Let  $\bar{\rho} > 0$  and  $\bar{\mathcal{T}}$  be two constant such that (1.6) is fulfilled. Suppose that  $d \geq 3$ , and that  $p$  satisfies*

$$(1.8) \quad 2 \leq p < d \quad \text{and} \quad p \leq 2d/(d-2).$$

*There exists a constant  $c = c(p, d, \lambda, \mu, P, \kappa, C_v, \bar{\rho}, \bar{\mathcal{T}})$  such that if  $a_0 \triangleq \rho_0 - \bar{\rho}$  is in  $\dot{B}_{p,1}^{\frac{d}{p}}$ , if  $v_0 \triangleq u_0$  is in  $\dot{B}_{p,1}^{\frac{d}{p}-1}$ , if  $\theta_0 \triangleq \mathcal{T}_0 - \bar{\mathcal{T}}$  is in  $\dot{B}_{p,1}^{\frac{d}{p}-2}$  and if in addition the low frequency part<sup>2</sup>  $(a_0^\ell, v_0^\ell, \theta_0^\ell)$  of  $(a_0, v_0, \theta_0)$  is in  $\dot{B}_{2,1}^{\frac{d}{2}-1}$  with*

$$(1.9) \quad \mathcal{X}_{p,0} \triangleq \|(a_0, v_0, \theta_0)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^\ell + \|a_0\|_{\dot{B}_{p,1}^{\frac{d}{p}}}^h + \|v_0\|_{\dot{B}_{p,1}^{\frac{d}{p}-1}}^h + \|\theta_0\|_{\dot{B}_{p,1}^{\frac{d}{p}-2}}^h \leq c$$

*then System (1.1)-(1.5) supplemented with the initial condition*

$$(1.10) \quad (\rho, u, \mathcal{T})|_{t=0} = (\rho_0, u_0, \mathcal{T}_0)$$

*admits a unique global-in-time solution  $(\rho, u, \mathcal{T})$  with  $\rho = \bar{\rho} + a$ ,  $u = v$  and  $\mathcal{T} = \bar{\mathcal{T}} + \theta$ , where  $(a, v, \theta)$  belongs to the space  $X_p$  defined by:*

$$(a, v, \theta)^\ell \in \tilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1}) \cap L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1}), \quad a^h \in \tilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}_{p,1}^{\frac{d}{p}}) \cap L^1(\mathbb{R}_+; \dot{B}_{p,1}^{\frac{d}{p}}),$$

$$v^h \in \tilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}_{p,1}^{\frac{d}{p}-1}) \cap L^1(\mathbb{R}_+; \dot{B}_{p,1}^{\frac{d}{p}+1}), \quad \theta^h \in \tilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}_{p,1}^{\frac{d}{p}-2}) \cap L^1(\mathbb{R}_+; \dot{B}_{p,1}^{\frac{d}{p}}).$$

*Moreover, we have for some constant  $C = C(p, d, \lambda, \mu, P, \kappa, C_v, \bar{\rho}, \bar{\mathcal{T}})$ ,*

$$(1.11) \quad \mathcal{X}_p(t) \leq C \mathcal{X}_{p,0},$$

*for any  $t > 0$ , where*

$$(1.12) \quad \mathcal{X}_p(t) \triangleq \|(a, v, \theta)\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})}^\ell + \|(a, v, \theta)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell$$

$$+ \|a\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}}) \cap L_t^1(\dot{B}_{p,1}^{\frac{d}{p}+1})}^h + \|v\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1}) \cap L_t^1(\dot{B}_{p,1}^{\frac{d}{p}+1})}^h + \|\theta\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-2}) \cap L_t^1(\dot{B}_{p,1}^{\frac{d}{p}})}^h.$$

The natural next step is to look for a more accurate description of the long time behavior of the solutions. Recall that Matsumura and Nishida in [29] proved that if the initial data are a small perturbation in  $H^3(\mathbb{R}^3) \times L^1(\mathbb{R}^3)$  of  $(\bar{\rho}, 0, \bar{\mathcal{T}})$  then<sup>3</sup>

$$(1.13) \quad \sup_{t \geq 0} \langle t \rangle^{\frac{3}{4}} \|(\rho - \bar{\rho}, u, \mathcal{T} - \bar{\mathcal{T}})(t)\|_{L^2} < \infty, \quad \text{with} \quad \langle t \rangle \triangleq \sqrt{1 + t^2}.$$

It turns out that the above behavior is kind of universal: some years latter Kawashima in [22] exhibited similar decay rates for hyperbolic-parabolic composite systems satisfying what is now called the “Shizuta-Kawashima” stability criterion.

Still in the framework of solutions with high Sobolev regularity, there are lots of recent improvements concerning the large time description of solutions to the compressible Navier-Stokes equations. In particular, some informations are now available on the wave aspect of the solutions. In one dimension space and in the isentropic case, Zeng [36] showed

<sup>2</sup>See the definitions in (3.3) and just below (3.7).

<sup>3</sup>Similar decay rates have been established in the half-space or exterior domain cases, see for example [23, 24, 31].

the  $L^1$  convergence to the nonlinear Burgers' diffusive wave. For multi-dimensional diffusion waves, Hoff and Zumbrun [19, 20] gave a detailed analysis for the Green's function and derived the  $L^\infty$  time-decay rates of diffusive waves. In [28], Liu and Wang exhibited pointwise estimates of diffusion waves with the optimal time-decay rate in odd dimension, that corresponds to the *weak Huygens' principle*. This was generalized later to the full system (1.1) in [26]. Recently, Liu & Noh [27] provided an exhaustive classification of the different type of waves in the long-time asymptotics as a combination of low frequency waves, the dissipative Huygens, diffusion and Riesz waves.

In the present paper, we aim at proving optimal time-decay estimates for (1.1) *within* the critical regularity framework of Theorem 1.1. Recall that in the (simpler) barotropic case, Okita [32] proved the optimal  $L^2$  decay rate in dimension  $d \geq 3$ . The first author [12] proposed another description which allows to deal with the case  $d = 2$  in the  $L^2$  critical framework. Very recently, in [14] we improved the approach in [12] and succeeded in establishing optimal decay estimates in the general  $L^p$  critical framework for all dimensions  $d \geq 2$ . As a first attempt of generalization, we here establish similar results for the full Navier-Stokes equations (1.1). In fact, thanks to an improvement of our method, we shall obtain what we believe to be the optimal decay exponents for the full nonlinear system.

## 2. REFORMULATION OF OUR PROBLEM AND MAIN RESULTS

Let us assume that the density and the temperature tend to some positive constants  $\bar{\rho}$  and  $\bar{\mathcal{T}}$ , at infinity. Setting  $\mathcal{A} \triangleq \mu\Delta + (\lambda + \mu)\nabla\text{div}$ ,  $\rho = \bar{\rho}(1 + b)$  and  $\mathcal{T} = \bar{\mathcal{T}} + \mathfrak{T}$ , we see from (1.1) and (1.5) that, whenever  $b > -1$ , the triplet  $(b, u, \theta)$  satisfies<sup>4</sup>

$$\begin{cases} \partial_t b + u \cdot \nabla b + (1 + b)\text{div} u = 0, \\ \partial_t u + u \cdot \nabla u - \frac{\mathcal{A}u}{\bar{\rho}(1+b)} + \frac{\pi'_0(\bar{\rho}(1+b))}{1+b}\nabla b + \frac{\pi_1(\bar{\rho}(1+b))}{\bar{\rho}(1+b)}\nabla \mathfrak{T} + \frac{\pi'_1(\bar{\rho}(1+b))}{1+b}\mathfrak{T}\nabla b = 0, \\ \partial_t \mathfrak{T} + u \cdot \nabla \mathfrak{T} + (\bar{\mathcal{T}} + \mathfrak{T})\frac{\pi_1(\bar{\rho}(1+b))}{\bar{\rho}C_v(1+b)}\text{div} u - \frac{\kappa}{\bar{\rho}C_v(1+b)}\Delta \mathfrak{T} = \frac{2\mu D(u):D(u) + \lambda(\text{div} u)^2}{\bar{\rho}C_v(1+b)}. \end{cases}$$

Then, setting  $\nu \triangleq \lambda + 2\mu$ ,  $\bar{\nu} \triangleq \nu/\bar{\rho}$ ,  $\chi_0 \triangleq \partial_\rho P(\bar{\rho}, \bar{\mathcal{T}})^{-\frac{1}{2}}$ , and performing the change of unknowns

$$a(t, x) = b(\bar{\nu}\chi_0^2 t, \bar{\nu}\chi_0 x), \quad v(t, x) = \chi_0 u(\bar{\nu}\chi_0^2 t, \bar{\nu}\chi_0 x), \quad \theta(t, x) = \chi_0 \sqrt{\frac{C_v}{\bar{\mathcal{T}}}} \mathfrak{T}(\bar{\nu}\chi_0^2 t, \bar{\nu}\chi_0 x),$$

we finally obtain

$$(2.1) \quad \begin{cases} \partial_t a + \text{div} v = f, \\ \partial_t v - \tilde{\mathcal{A}}v + \nabla a + \gamma \nabla \theta = g, \\ \partial_t \theta - \beta \Delta \theta + \gamma \text{div} v = k, \end{cases}$$

with

$$\tilde{\mathcal{A}} \triangleq \frac{\mathcal{A}}{\nu}, \quad \beta \triangleq \frac{\kappa}{\nu C_v}, \quad \gamma = \frac{\chi_0}{\bar{\rho}} \sqrt{\frac{\bar{\mathcal{T}}}{C_v}} \pi_1(\bar{\rho}),$$

and where the nonlinear terms  $f$ ,  $g$  and  $k$  are given by

$$\begin{aligned} f &\triangleq -\text{div}(av), \quad g \triangleq -v \cdot \nabla v - I(a)\tilde{\mathcal{A}}v - K_1(a)\nabla a - K_2(a)\nabla \theta - \theta \nabla K_3(a) \\ \text{and } k &\triangleq -v \cdot \nabla \theta - \beta I(a)\Delta \theta + \frac{Q(\nabla v, \nabla v)}{1+a} - (\tilde{K}_1(a) + \tilde{K}_2(a)\theta)\text{div} v \end{aligned}$$

<sup>4</sup>For better readability, from now on, we just denote  $D_x$  by  $D$ ,  $\text{div}_x$  by  $\text{div}$ , and so on.

with

$$\begin{aligned}
I(a) &\triangleq \frac{a}{1+a}, \quad K_1(a) \triangleq \frac{\partial_\rho P(\bar{\rho}(1+a), \bar{T})}{(1+a)\partial_\rho P(\bar{\rho}, \bar{T})} - 1, \quad K_2(a) \triangleq \frac{\chi_0}{\bar{\rho}} \sqrt{\frac{\bar{T}}{C_v}} \left( \frac{\pi_1(\bar{\rho}(1+a))}{1+a} - \pi_1(\bar{\rho}) \right), \\
K_3(a) &\triangleq \chi_0 \sqrt{\frac{\bar{T}}{C_v}} \int_0^a \frac{\pi'_1(\bar{\rho}(1+z))}{1+z} dz, \quad \tilde{K}_2(a) \triangleq \frac{\pi_1(\bar{\rho}(1+a))}{C_v \bar{\rho}(1+a)}, \\
\tilde{K}_1(a) &\triangleq \frac{\chi_0}{\bar{\rho}} \sqrt{\frac{\bar{T}}{C_v}} \left( \frac{\pi_1(\bar{\rho}(1+a))}{1+a} - \pi_1(\bar{\rho}) \right), \\
Q(A, B) &\triangleq \frac{1}{\nu \chi_0} \sqrt{\frac{1}{\bar{T} C_v}} (2\mu A : B + \lambda \operatorname{Tr} A \operatorname{Tr} B).
\end{aligned}$$

In fact, the exact value of  $K_1, K_2, K_3, \tilde{K}_1$  and  $\tilde{K}_2$  will not matter in our analysis. We shall just use that those functions are smooth and that  $K_1(0) = K_2(0) = K_3(0) = \tilde{K}_1(0) = 0$ .

One can now state the main result of the paper.

**Theorem 2.1.** *Let the assumptions of Theorem 1.1 be in force and  $(a, v, \theta)$  be the corresponding global solution. Let the real number  $s_1$  satisfy*

$$(2.2) \quad \max\left(0, 2 - \frac{d}{2}\right) \leq s_1 \leq s_0$$

with  $s_0 \triangleq \frac{2d}{p} - \frac{d}{2}$ . There exists a constant  $c > 0$  depending only on  $p, d, \lambda, \mu, P, \kappa, C_v, \bar{\rho}, \bar{T}$  such that if

$$(2.3) \quad \mathcal{D}_{p,0} \triangleq \|(a_0, v_0, \theta_0)\|_{\dot{B}_{2,\infty}^{-s_1}}^\ell \leq c,$$

then for all small enough  $\varepsilon > 0$ , we have

$$(2.4) \quad \mathcal{D}_p(t) \lesssim (\mathcal{D}_{p,0} + \|(\nabla a_0, v_0)\|_{\dot{B}_{p,1}^{\frac{d}{p}-1}}^h + \|\theta_0\|_{\dot{B}_{p,1}^{\frac{d}{p}-2}}^h) \quad \text{for all } t \geq 0,$$

where, setting  $\alpha \triangleq s_1 + \frac{d}{2} + \frac{1}{2} - \varepsilon$ , the norm  $\mathcal{D}_p(t)$  is defined by

$$\begin{aligned}
(2.5) \quad \mathcal{D}_p(t) &\triangleq \sup_{s \in [\varepsilon - s_1, \frac{d}{2} + 1]} \|\langle \tau \rangle^{\frac{s_1+s}{2}} (a, v, \theta)\|_{L_t^\infty(\dot{B}_{2,1}^s)}^\ell \\
&+ \|\langle \tau \rangle^\alpha a\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}})}^h + \|\langle \tau \rangle^\alpha v\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})}^h + \|\langle \tau \rangle^\alpha \theta\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h + \|\tau^\alpha (\nabla v, \theta)\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}})}^h.
\end{aligned}$$

The above statement deserves some comments:

- A similar result may be proved if the physical coefficients  $\lambda, \mu$  and  $\kappa$  depend smoothly on the density. Here, we took them constant to avoid more technicalities.
- To the best of our knowledge, whether well-posedness holds true in critical spaces for the full Navier-Stokes equations in  $\mathbb{R}^2$  is an open question. At the same time, it has been proved for slightly more regular data (see [10, 11]) and we believe our approach to yield decay estimates for the corresponding solutions (computations are expected to be wilder, though).
- For  $p = 2$  and  $s_1 = \frac{d}{2}$ , hypothesis (2.3) is less restrictive than the standard  $L^1$  condition first because it only concerns the low frequencies and, second, because we have  $L^r \hookrightarrow \dot{B}_{2,\infty}^{-s_1}$  with  $\frac{1}{r} = \frac{1}{2} + \frac{s_1}{d}$ . A similar assumption (for all frequencies and for  $p = 2$  and  $s_1 = \frac{d}{2}$ ) appears in several recent results : for the Boltzmann equation in the work by Sohinger and Strain [33], and for hyperbolic systems with dissipation in the joint paper of Kawashima with the second author [34, 35]. One

can also mention the work by Guo and Wang [15] that replaces the  $L^1$  assumption by homogeneous Sobolev norms of negative order.

- Compared to the barotropic case studied in [14], the decay functional  $\mathcal{D}_p$  contains an additional decay information on the temperature. Furthermore, we improved the decay exponent for high frequencies : in the case  $s_1 = s_0 \triangleq \frac{2d}{p} - \frac{d}{2}$  (which is the only one that has been considered in [14]), we get  $\alpha = \frac{2d}{p} + \frac{1}{2} - \varepsilon$  instead of 1. We believe exponent  $\frac{s_1}{2} + \frac{d}{2} + \frac{1}{2}$  as well as the upper bound  $s \leq \frac{d}{2} + 1$  for the first term of  $\mathcal{D}_p$  to be optimal (see the very end of the present paper for more explanations).
- If we replace (2.3) with the following slightly stronger hypothesis:

$$\|(a_0, v_0, \theta_0)\|_{\dot{B}_{2,1}^{-s_1}}^\ell \ll 1$$

then one can take  $\varepsilon = 0$  in the definitions of  $\mathcal{D}_p$  and  $\alpha$ .

- We expect to have similar decay estimates for  $s_1$  belonging to the whole range  $(1 - \frac{d}{2}, \frac{2d}{p} - \frac{d}{2}]$ . As the analysis for  $s_1 < \max(0, 2 - \frac{d}{2})$  is much more technical (it relies on tricky product estimates in Besov spaces), we chose to present only the case (2.2) which is already more general than that in [14].

From Theorem 2.1 and standard embedding, we readily get the following algebraic decay rates for the  $L^p$  norms of the solution:

**Corollary 2.1.** *Denote  $\Lambda^s f \triangleq \mathcal{F}^{-1}(|\cdot|^s \mathcal{F} f)$ . The solution of Theorem 1.1 satisfies*

$$\|\Lambda^s(\rho - \bar{\rho})\|_{L^p} \lesssim (\mathcal{D}_{p,0} + \|(\nabla a_0, v_0)\|_{\dot{B}_{p,1}^{\frac{d}{p}-1}}^h + \|\theta_0\|_{\dot{B}_{p,1}^{\frac{d}{p}-2}}^h) \langle t \rangle^{-\frac{s_1+s}{2}} \quad \text{if } -s_1 < s \leq \frac{d}{p},$$

$$\|\Lambda^s u\|_{L^p} \lesssim (\mathcal{D}_{p,0} + \|(\nabla a_0, v_0)\|_{\dot{B}_{p,1}^{\frac{d}{p}-1}}^h + \|\theta_0\|_{\dot{B}_{p,1}^{\frac{d}{p}-2}}^h) \langle t \rangle^{-\frac{s_1+s}{2}} \quad \text{if } -s_1 < s \leq \frac{d}{p} - 1,$$

$$\|\Lambda^s(\mathcal{T} - \bar{\mathcal{T}})\|_{L^p} \lesssim (\mathcal{D}_{p,0} + \|(\nabla a_0, v_0)\|_{\dot{B}_{p,1}^{\frac{d}{p}-1}}^h + \|\theta_0\|_{\dot{B}_{p,1}^{\frac{d}{p}-2}}^h) \langle t \rangle^{-\frac{s_1+s}{2}} \quad \text{if } -s_1 < s \leq \frac{d}{p} - 2.$$

**Remark 2.1.** *Taking  $p = 2$ ,  $s_1 = d/2$  and  $s = 0$  in Corollary 2.1 leads back to the standard optimal  $L^1$ - $L^2$  decay rates in (1.13), but for much less regular global solutions. Also, note that the derivative index can take both negative and nonnegative values rather than nonnegative integers only, and our results can thus be regarded as the natural extension of the classical results of [30].*

Let us briefly present the strategy for proving Theorem 2.1. The usual approach to get decay estimates of the type (1.13) is to take advantage of  $L^1$ - $L^2$  decay estimates for the linear system corresponding to the left-hand side of (2.1), treating the nonlinear right-hand side  $(f, g, k)$  by means of Duhamel formula (see [22, 30] and references therein). That basic argument fails in the critical regularity spaces, though, as one cannot afford any *loss of regularity* for the high frequency part of the solution (for example,  $u \cdot \nabla a$  induces a loss of one derivative, while there is no smoothing for  $a$ , solution of a transport equation). Furthermore, the standard approach completely ignores the fact that the semi-group associated to (2.1) behaves differently for low and high frequencies. As regards the low frequency part of the solution, the linearized equations behave like the heat equation (at least in a  $L^2$  type framework) and it is possible to adapt the standard method relying on  $L^1$ - $L^2$  estimate. The only difference is that owing to our  $L^p$  type assumption on the high frequencies and our more general low frequency assumption (2.3), the quadratic terms

are in  $L^r$  with  $\frac{1}{r} = \frac{1}{2} + \frac{s_1}{d}$ , and it is thus natural to resort to  $L^r$ - $L^2$  estimates (or sometimes to  $\dot{B}_{2,\infty}^{-s_1}$ - $L^2$  ones) rather than  $L^1$ - $L^2$  estimates.

We proceed differently for the analysis of the high frequencies decay of the solution. The idea is to work with a so-called “effective velocity”  $w$  (introduced by D. Hoff in [18] and first used in the context of critical regularity by B. Haspot in [17]) such that, up to low order terms, the divergence-free part of  $u$ , the temperature  $\theta$  and  $w$  fulfill a parabolic system while  $a$  satisfies a damped transport equation. Performing  $L^p$  estimates directly on that system after localization in the Fourier space and using suitable commutator estimates to handle the convection term in the density equation, it is possible to eventually get *optimal decay exponents* for high frequencies.

The rest of the paper unfolds as follows. In Section 3, we introduce some notation, recall basic results concerning Besov spaces, paradifferential calculus, product and commutator estimates. Section 4 is devoted to the proof of Theorem 2.1 and of Corollary 2.1.

### 3. NOTATIONS, FUNCTIONAL SPACES AND BASIC TOOLS

Throughout the paper,  $C$  stands for a harmless positive “constant”, the meaning of which is clear from the context, and we sometimes write  $f \lesssim g$  instead of  $f \leq Cg$ . The notation  $f \approx g$  means that  $f \lesssim g$  and  $g \lesssim f$ . For any Banach space  $X$  and  $f, g \in X$ , we agree that  $\|(f, g)\|_X \triangleq \|f\|_X + \|g\|_X$ . Finally, for all  $T > 0$  and  $\rho \in [1, +\infty]$ , we denote by  $L_T^\rho(X) \triangleq L^\infty([0, T]; X)$  the set of measurable functions  $f : [0, T] \rightarrow X$  such that  $t \mapsto \|f(t)\|_X$  is in  $L^\rho(0, T)$ .

Let us next recall the definition and a few basic properties of Besov spaces (more details may be found in e.g. Chap. 2 and 3 of [1]). We start with a dyadic decomposition in Fourier variables: fix some smooth radial non increasing function  $\chi$  supported in  $B(0, \frac{4}{3})$  and with value 1 on  $B(0, \frac{3}{4})$ , then set  $\varphi(\xi) = \chi(\xi/2) - \chi(\xi)$  so that

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j} \cdot) = 1 \text{ in } \mathbb{R}^d \setminus \{0\} \quad \text{and} \quad \text{Supp } \varphi \subset \{\xi \in \mathbb{R}^d : 3/4 \leq |\xi| \leq 8/3\}.$$

The homogeneous dyadic blocks  $\dot{\Delta}_j$  are defined by

$$\dot{\Delta}_j f \triangleq \varphi(2^{-j} D) f = \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) \mathcal{F} f) = 2^{jd} h(2^j \cdot) \star f \quad \text{with} \quad h \triangleq \mathcal{F}^{-1} \varphi.$$

The *Littlewood-Paley decomposition* of a general tempered distribution  $f$  reads

$$(3.1) \quad f = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j f.$$

That equality holds true in the tempered distribution meaning, if  $f$  satisfies

$$(3.2) \quad \lim_{j \rightarrow -\infty} \|\dot{S}_j f\|_{L^\infty} = 0,$$

where  $\dot{S}_j$  stands for the low frequency cut-off  $\dot{S}_j \triangleq \chi(2^{-j} D)$ .

In many parts of the paper, we use the notation  $z^\ell$  and  $z^h$  with

$$(3.3) \quad z^\ell := \dot{S}_{j_0} z \quad \text{and} \quad z^h := (\text{Id} - \dot{S}_{j_0}) z,$$

where the value of the parameter  $j_0$  depends on the coefficients of (2.1) through the proof of the main theorem.

Next, we come to the definition of the homogeneous Besov spaces.

**Definition 3.1.** For  $s \in \mathbb{R}$  and  $1 \leq p, r \leq \infty$ , the homogeneous Besov space  $\dot{B}_{p,r}^s$  is the set of tempered distributions  $f$  satisfying (3.2) and

$$\|f\|_{\dot{B}_{p,r}^s} \triangleq \left\| (2^{js} \|\dot{\Delta}_j f\|_{L^p}) \right\|_{\ell^r(\mathbb{Z})} < \infty.$$

In order to state optimal regularity estimates for the heat equation, we need the following semi-norms first introduced by J.-Y. Chemin in [5] for all  $0 \leq T \leq +\infty$ ,  $s \in \mathbb{R}$  and  $1 \leq r, p, \varrho \leq \infty$ :

$$(3.4) \quad \|f\|_{\tilde{L}_T^\varrho(\dot{B}_{p,r}^s)} \triangleq \left\| (2^{js} \|\dot{\Delta}_j f\|_{L_T^\varrho(L^p)}) \right\|_{\ell^r(\mathbb{Z})}.$$

Index  $T$  will be sometimes omitted if  $T = +\infty$ , and we denote

$$(3.5) \quad \tilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}_{p,r}^s) \triangleq \{f \in \mathcal{C}(\mathbb{R}_+; \dot{B}_{p,r}^s) \text{ s.t. } \|f\|_{\tilde{L}^\infty(\dot{B}_{p,r}^s)} < \infty\}.$$

Recall (see e.g. [1]) the following optimal regularity estimates for the heat equation:

**Proposition 3.1.** Let  $\sigma \in \mathbb{R}$ ,  $(p, r) \in [1, \infty]^2$  and  $1 \leq \rho_2 \leq \rho_1 \leq \infty$ . Let  $u$  satisfy

$$\begin{cases} \partial_t u - \mu \Delta u = f, \\ u|_{t=0} = u_0. \end{cases}$$

Then for all  $T > 0$  the following a priori estimate is fulfilled:

$$(3.6) \quad \mu^{\frac{1}{\rho_1}} \|u\|_{\tilde{L}_T^{\rho_1}(\dot{B}_{p,r}^{\sigma+\frac{2}{\rho_1}})} \lesssim \|u_0\|_{\dot{B}_{p,r}^\sigma} + \mu^{\frac{1}{\rho_2}-1} \|f\|_{\tilde{L}_T^{\rho_2}(\dot{B}_{p,r}^{\sigma-2+\frac{2}{\rho_2}})}.$$

The same estimate holds true (with a different dependency with respect to the viscosity coefficients) for the solutions to the following Lamé system

$$\begin{cases} \partial_t u - \mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u = f, \\ u|_{t=0} = u_0, \end{cases}$$

whenever  $\mu > 0$  and  $\lambda + 2\mu > 0$ .

The properties of continuity for the product, commutators and composition involving standard Besov norms remain true when using the semi-norms defined in (3.4). The general principle is that the time Lebesgue exponent has to be treated according to Hölder inequality. Furthermore, Minkowski's inequality allows to compare  $\|\cdot\|_{\tilde{L}_T^\varrho(\dot{B}_{p,r}^s)}$  with the more standard Lebesgue-Besov semi-norms of  $L_T^\varrho(\dot{B}_{p,r}^s)$  as follows:

$$(3.7) \quad \|f\|_{\tilde{L}_T^\varrho(\dot{B}_{p,r}^s)} \leq \|f\|_{L_T^\varrho(\dot{B}_{p,r}^s)} \text{ if } r \geq \varrho, \quad \|f\|_{\tilde{L}_T^\varrho(\dot{B}_{p,r}^s)} \geq \|f\|_{L_T^\varrho(\dot{B}_{p,r}^s)} \text{ if } r \leq \varrho.$$

Restricting Besov norms to the low or high frequencies parts of distributions plays a fundamental role in our approach. For that reason, we shall often use the following notation for some suitable integer  $j_0$ <sup>5</sup>:

$$\begin{aligned} \|f\|_{\dot{B}_{p,1}^s}^\ell &\triangleq \sum_{j \leq j_0} 2^{js} \|\dot{\Delta}_j f\|_{L^p} \quad \text{and} \quad \|f\|_{\dot{B}_{p,1}^s}^h \triangleq \sum_{j \geq j_0-1} 2^{js} \|\dot{\Delta}_j f\|_{L^p}, \\ \|f\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^s)}^\ell &\triangleq \sum_{j \leq j_0} 2^{js} \|\dot{\Delta}_j f\|_{L_T^\infty(L^p)} \quad \text{and} \quad \|f\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^s)}^h \triangleq \sum_{j \geq j_0-1} 2^{js} \|\dot{\Delta}_j f\|_{L_T^\infty(L^p)}. \end{aligned}$$

We shall use the following nonlinear generalization of the Bernstein inequality (see e.g. Lemma 8 in [9]) that will be the key to controlling the  $L^p$  norms of the solution to the spectrally localized system (2.1):

<sup>5</sup>For technical reasons, it is convenient to have a small overlap between low and high frequencies.



**Proposition 3.2.** *There exists  $c$  depending only on  $d$ ,  $R_1$  and  $R_2$  so that if*

$$(3.8) \quad \text{Supp } \mathcal{F}f \subset \{\xi \in \mathbb{R}^d : R_1\lambda \leq |\xi| \leq R_2\lambda\},$$

*is fulfilled then for all  $1 < p < \infty$ ,*

$$(3.9) \quad c\lambda^2 \left( \frac{p-1}{p} \right) \int_{\mathbb{R}^d} |f|^p dx \leq (p-1) \int_{\mathbb{R}^d} |\nabla f|^2 |f|^{p-2} dx = - \int_{\mathbb{R}^d} \Delta f |f|^{p-2} f dx.$$

Below are embedding properties which are used several times:

**Lemma 3.1.** • *For any  $p \in [1, \infty]$  we have the continuous embedding*

$$\dot{B}_{p,1}^0 \hookrightarrow L^p \hookrightarrow \dot{B}_{p,\infty}^0.$$

- *If  $s \in \mathbb{R}$ ,  $1 \leq p_1 \leq p_2 \leq \infty$  and  $1 \leq r_1 \leq r_2 \leq \infty$ , then  $\dot{B}_{p_1,r_1}^s \hookrightarrow \dot{B}_{p_2,r_2}^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}$ .*
- *The space  $\dot{B}_{p,1}^{\frac{d}{p}}$  is continuously embedded in the set of bounded continuous functions (going to 0 at infinity if  $p < \infty$ ).*

Let us also mention the following interpolation inequality

**Lemma 3.2.** *Let  $1 \leq p, r_1, r_2, r \leq \infty$ ,  $\sigma_1 \neq \sigma_2$  and  $\theta \in (0, 1)$ . Then*

$$(3.10) \quad \|f\|_{\dot{B}_{p,r}^{\theta\sigma_2+(1-\theta)\sigma_1}} \lesssim \|f\|_{\dot{B}_{p,r_1}^{\sigma_1}}^{1-\theta} \|f\|_{\dot{B}_{p,r_2}^{\sigma_2}}^{\theta}.$$

As already pointed out, the low frequency hypothesis is less restrictive than the usual  $L^q$ . This is a consequence of the following embedding:

**Lemma 3.3.** *Suppose that  $\sigma > 0$  and  $1 \leq q < 2$ . One has*

$$(3.11) \quad \|f\|_{\dot{B}_{r,\infty}^{-\sigma}} \lesssim \|f\|_{L^q} \quad \text{with} \quad \frac{1}{q} - \frac{1}{r} = \frac{\sigma}{d}.$$

The following product laws and commutator estimates proved in e.g. [1] and [14] will play a fundamental role in the analysis of the bilinear terms of (1.7).

**Proposition 3.3.** *Let  $\sigma > 0$  and  $1 \leq p, r \leq \infty$ . Then  $\dot{B}_{p,r}^{\sigma} \cap L^{\infty}$  is an algebra and*

$$(3.12) \quad \|fg\|_{\dot{B}_{p,r}^{\sigma}} \lesssim \|f\|_{L^{\infty}} \|g\|_{\dot{B}_{p,r}^{\sigma}} + \|g\|_{L^{\infty}} \|f\|_{\dot{B}_{p,r}^{\sigma}}.$$

*Let the real numbers  $\sigma_1, \sigma_2, p_1$  and  $p_2$  be such that*

$$\sigma_1 + \sigma_2 > 0, \quad \sigma_1 \leq \frac{d}{p_1}, \quad \sigma_2 \leq \frac{d}{p_2}, \quad \sigma_1 \geq \sigma_2, \quad \frac{1}{p_1} + \frac{1}{p_2} \leq 1.$$

*Then we have*

$$(3.13) \quad \|fg\|_{\dot{B}_{q,1}^{\sigma_2}} \lesssim \|f\|_{\dot{B}_{p_1,1}^{\sigma_1}} \|g\|_{\dot{B}_{p_2,1}^{\sigma_2}} \quad \text{with} \quad \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\sigma_1}{d}.$$

*Finally, for exponents  $\sigma > 0$  and  $1 \leq p_1, p_2, q \leq \infty$  satisfying*

$$\frac{d}{p_1} + \frac{d}{p_2} - d \leq \sigma \leq \min\left(\frac{d}{p_1}, \frac{d}{p_2}\right) \quad \text{and} \quad \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\sigma}{d},$$

*we have*

$$(3.14) \quad \|fg\|_{\dot{B}_{q,\infty}^{-\sigma}} \lesssim \|f\|_{\dot{B}_{p_1,1}^{\sigma}} \|g\|_{\dot{B}_{p_2,\infty}^{-\sigma}}.$$

Proposition 3.3 is not enough to handle the case  $2p > d$  in the proof of Theorem 2.1. We shall make use of the following estimates proved recently in [14].

**Proposition 3.4.** *Let  $j_0 \in \mathbb{Z}$ , and denote  $z^\ell \triangleq \dot{S}_{j_0} z$ ,  $z^h \triangleq z - z^\ell$  and, for any  $s \in \mathbb{R}$ ,*

$$\|z\|_{\dot{B}_{2,\infty}^s}^\ell \triangleq \sup_{j \leq j_0} 2^{js} \|\dot{\Delta}_j z\|_{L^2}.$$

*There exists a universal integer  $N_0$  such that for any  $2 \leq p \leq 4$  and  $\sigma > 0$ , we have*

$$(3.15) \quad \|fg^h\|_{\dot{B}_{2,\infty}^{-s_0}}^\ell \leq C(\|f\|_{\dot{B}_{p,1}^\sigma} + \|\dot{S}_{j_0+N_0} f\|_{L^{p^*}}) \|g^h\|_{\dot{B}_{p,\infty}^{-\sigma}}$$

$$(3.16) \quad \|f^h g\|_{\dot{B}_{2,\infty}^{-s_0}}^\ell \leq C(\|f^h\|_{\dot{B}_{p,1}^\sigma} + \|\dot{S}_{j_0+N_0} f^h\|_{L^{p^*}}) \|g\|_{\dot{B}_{p,\infty}^{-\sigma}}$$

*with  $s_0 \triangleq \frac{2d}{p} - \frac{d}{2}$  and  $\frac{1}{p^*} \triangleq \frac{1}{2} - \frac{1}{p}$ , and  $C$  depending only on  $j_0$ ,  $d$  and  $\sigma$ .*

The terms in System (2.1) corresponding to the functions  $K_1, K_2, K_3, \tilde{K}_1$  and  $\tilde{K}_2$  will be bounded thanks to the following classical result:

**Proposition 3.5.** *Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be smooth with  $F(0) = 0$ . For all  $1 \leq p, r \leq \infty$  and  $\sigma > 0$  we have  $F(f) \in \dot{B}_{p,r}^\sigma \cap L^\infty$  for  $f \in \dot{B}_{p,r}^\sigma \cap L^\infty$ , and*

$$(3.17) \quad \|F(f)\|_{\dot{B}_{p,r}^\sigma} \leq C \|f\|_{\dot{B}_{p,r}^\sigma}$$

*with  $C$  depending only on  $\|f\|_{L^\infty}$ ,  $F'$  (and higher derivatives),  $\sigma$ ,  $p$  and  $d$ .*

**Remark 3.1.** *As Theorem 1.1 involves  $\tilde{L}_T^g(\dot{B}_{p,r}^s)$  semi-norms, we shall very often use Propositions 3.3, 3.4 and 3.5 adapted to those spaces. The general rule is that exactly the same estimates hold true, once the time Lebesgue exponents have been treated according to Hölder inequality.*

#### 4. THE PROOF OF DECAY ESTIMATES

In this section, we prove Theorem 2.1. We start with the global solution  $(a, v, \theta)$  given by Theorem 1.1, that satisfies (1.11) and thus in particular

$$(4.1) \quad \|a\|_{\tilde{L}^\infty(\dot{B}_{p,1}^{\frac{d}{p}})} \ll 1.$$

Then there only remains to prove (2.4). The overall strategy is to combine frequency-localization of the linear part of the full system with the Duhamel principle to handle the nonlinear terms.

We shall proceed in three steps. Step 1 is dedicated to the proof of decay estimates for the low frequency part of  $(a, v, \theta)$ , that is

$$\mathcal{D}_{p,1}(t) \triangleq \sup_{s \in [\varepsilon - s_1, \frac{d}{2} + 1]} \|\langle \tau \rangle^{\frac{s_1+s}{2}}(a, v, \theta)\|_{L_t^\infty(\dot{B}_{2,1}^s)}^\ell.$$

To this end, we shall exhibit the low frequency decay exponents for the linear system corresponding to the left-hand side of (2.1). This will be achieved by means of a simple energy method on the spectrally localized system, *without computing the Green function of the linear system*. Denoting by  $(E(t))_{t \geq 0}$  the corresponding semi-group, we shall get

$$(4.2) \quad \sup_{t \geq 0} \langle t \rangle^{\frac{s_1+s}{2}} \|E(t)U_0\|_{\dot{B}_{2,1}^s}^\ell \lesssim \|U_0\|_{\dot{B}_{2,\infty}^{-s_1}}^\ell \quad \text{if } s > -s_1.$$

Then taking advantage of Duhamel formula reduces the proof to that of suitable decay estimates in the space  $\dot{B}_{2,\infty}^{-s_1}$  for all the nonlinear terms in  $f$ ,  $g$  and  $h$ . Using the embedding  $L^r \hookrightarrow \dot{B}_{2,\infty}^{-s_1}$  with  $\frac{1}{r} = \frac{s_1}{d} + \frac{1}{2}$  and the obvious product law  $L^{2r} \times L^{2r} \rightarrow L^r$ , it turns out to be (almost) enough to exhibit appropriate bounds for the solution in  $L^{2r}$ . In fact, that strategy fails only if  $p > \frac{d}{2}$ , owing to the term  $I(a)\Delta\theta^h$  which, according to Theorem 1.1,

is only in  $\dot{B}_{p,1}^{\frac{d}{p}-2}$  for a.a.  $t \geq 0$ . Clearly, if  $\frac{d}{p} - 2 < 0$  then one cannot expect that term to be in any Lebesgue space. To overcome that difficulty, we will have to use the more elaborate product laws in Besov spaces stated in Proposition 3.4.

Step 2 is devoted to bounding<sup>6</sup>

$$\mathcal{D}_{p,2}(t) \triangleq \|\langle \tau \rangle^\alpha a\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}})}^h + \|\langle \tau \rangle^\alpha v\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})}^h + \|\langle \tau \rangle^\alpha \theta\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h.$$

To this end, following Haspot's approach in [17], we introduce the *effective velocity*

$$(4.3) \quad w \triangleq \nabla(-\Delta)^{-1}(a - \operatorname{div} v).$$

Then, up to low order terms,  $w$ ,  $\theta$  and the divergence free part of the velocity satisfy a heat equation, while  $a$  fulfills a *damped* transport equation. That way of rewriting the full system allows to avoid the loss of one derivative arising from the convection term in the density equation. Basically, this is the same strategy as for the barotropic Navier-Stokes equations (see [14]). In the polytropic case however, one has to perform low and high frequency decompositions of the (new) nonlinear terms involving  $\theta$ , for  $\theta^h$  is less regular than  $v^h$  by one derivative.

In the last step, we establish a gain of regularity and decay altogether for the high frequencies of  $v$  and  $\theta$ , namely we bound

$$\mathcal{D}_{p,3}(t) \triangleq \|\tau^\alpha(\nabla v, \theta)\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}})}^h.$$

This strongly relies on the maximal regularity estimates for the heat and Lamé semi-groups (see Proposition 3.1). Compared to our previous paper [14], we found a way to improve substantially the decay rate exhibited in  $\mathcal{D}_{p,3}$ : we get  $t^{-\alpha}$  instead of just  $t^{-1}$ .

*Step 1: Bounds for the low frequencies.* Let  $(E(t))_{t \geq 0}$  be the semi-group associated to the left-hand side of (2.1). The standard Duhamel principle gives

$$(4.4) \quad \begin{pmatrix} a(t) \\ v(t) \\ \theta(t) \end{pmatrix} = E(t) \begin{pmatrix} a_0 \\ v_0 \\ \theta_0 \end{pmatrix} + \int_0^t E(t-\tau) \begin{pmatrix} f(\tau) \\ g(\tau) \\ k(\tau) \end{pmatrix} d\tau.$$

The following lemma states that the low frequency part of  $(a_L, v_L, \theta_L) \triangleq E(t)(a_0, v_0, \theta_0)$  behaves essentially like the solution to the heat equation.

**Lemma 4.1.** *Let  $(a_L, v_L, \theta_L)$  be the solution to the following system*

$$(4.5) \quad \begin{cases} \partial_t a_L + \operatorname{div} v_L = 0, \\ \partial_t v_L - \tilde{\mathcal{A}} v_L + \nabla a_L + \gamma \nabla \theta_L = 0, \\ \partial_t \theta_L - \beta \Delta \theta_L + \gamma \operatorname{div} v_L = 0, \end{cases}$$

*with the initial data*

$$(4.6) \quad (a_L, v_L, \theta_L)|_{t=0} = (a_0, v_0, \theta_0).$$

*Then, for any  $j_0 \in \mathbb{Z}$ , there exists a positive constant  $c_0 = c_0(\lambda, \mu, \beta, \gamma, j_0)$  such that*

$$(4.7) \quad \|(a_{L,j}, v_{L,j}, \theta_{L,j})(t)\|_{L^2} \lesssim e^{-c_0 2^{2j} t} \|(a_{0,j}, v_{0,j}, \theta_{0,j})\|_{L^2}$$

*for  $t \geq 0$  and  $j \leq j_0$ , where we set  $z_j = \Delta_j z$  for any  $z \in \mathcal{S}'(\mathbb{R}^d)$ .*

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<sup>6</sup>Recall that  $\alpha \triangleq s_1 + \frac{d}{2} + \frac{1}{2} - \varepsilon$ .

*Proof.* Set  $\omega_L \triangleq |D|^{-1} \operatorname{div} v_L$  with  $\mathcal{F}(|D|^s f)(\xi) \triangleq |\xi|^s \widehat{f}(\xi)$  ( $s \in \mathbb{R}$ ). Remembering that  $\mathcal{A} = \nu^{-1}(\mu\Delta + (\lambda + \mu)\nabla \operatorname{div})$  and that  $\nu = \lambda + 2\mu$ , we get

$$(4.8) \quad \begin{cases} \partial_t a_L + |D|\omega_L = 0, \\ \partial_t \omega_L - \Delta \omega_L - |D|a_L - \gamma|D|\theta_L = 0, \\ \partial_t \theta_L - \beta \Delta \theta_L + \gamma|D|\omega_L = 0. \end{cases}$$

Let  $(A, \Omega, \Theta)$  be the Fourier transform of  $(a_L, \omega_L, \theta_L)$ . Then, we have for all  $\varrho \triangleq |\xi|$ ,

$$(4.9) \quad \begin{cases} \partial_t A + \varrho \Omega = 0, \\ \partial_t \Omega + \rho^2 \Omega - \rho A - \gamma \rho \Theta = 0, \\ \partial_t \Theta + \beta \rho^2 \Theta + \gamma \rho \Omega = 0, \end{cases}$$

from which we get the following three identities:

$$(4.10) \quad \frac{1}{2} \frac{d}{dt} |A|^2 + \varrho \operatorname{Re}(A \bar{\Omega}) = 0,$$

$$(4.11) \quad \frac{1}{2} \frac{d}{dt} |\Omega|^2 + \varrho^2 |\Omega|^2 - \varrho \operatorname{Re}(A \bar{\Omega}) - \gamma \varrho \operatorname{Re}(\Theta \bar{\Omega}) = 0,$$

$$(4.12) \quad \frac{1}{2} \frac{d}{dt} |\Theta|^2 + \beta \varrho^2 |\Theta|^2 + \gamma \varrho \operatorname{Re}(\Theta \bar{\Omega}) = 0,$$

where  $\bar{f}$  indicates the complex conjugate of a function  $f$ .

By adding up (4.10), (4.11) and (4.12), we obtain

$$(4.13) \quad \frac{1}{2} \frac{d}{dt} |(A, \Omega, \Theta)|^2 + \tilde{\beta} \varrho^2 |(\Omega, \Theta)|^2 \leq 0 \quad \text{with } \tilde{\beta} \triangleq \min(1, \beta).$$

Next, to track the dissipation of  $A$ , we use the fact that

$$(4.14) \quad \frac{d}{dt} [-\operatorname{Re}(A \bar{\Omega})] + \varrho |A|^2 - \varrho |\Omega|^2 - \varrho^2 \operatorname{Re}(A \bar{\Omega}) + \gamma \varrho \operatorname{Re}(\Theta \bar{A}) = 0.$$

Combining with (4.10), we deduce that

$$(4.15) \quad \frac{1}{2} \frac{d}{dt} [\varrho |A|^2 - 2\varrho \operatorname{Re}(A \bar{\Omega})] + \varrho^2 |A|^2 - \varrho^2 |\Omega|^2 + \gamma \varrho^2 \operatorname{Re}(\Theta \bar{A}) = 0.$$

Therefore, introducing the “Lyapunov functional”

$$\mathcal{L}_\varrho^2(t) \triangleq |(A, \Omega, \Theta)|^2 + K(|\varrho A|^2 - 2\varrho \operatorname{Re}(A \bar{\Omega}))$$

for some  $K > 0$  (to be chosen hereafter), we get from (4.13) and (4.15) that

$$\frac{1}{2} \frac{d}{dt} \mathcal{L}_\varrho^2 + \varrho^2 (K|A|^2 + (\tilde{\beta} - K)|\Omega|^2 + \tilde{\beta}|\Theta|^2) + K\gamma \varrho^2 \operatorname{Re}(\Theta \bar{A}) \leq 0.$$

Then, taking advantage of the following Young inequality

$$|\gamma \operatorname{Re}(\Theta \bar{A})| \leq \frac{1}{2} |A|^2 + \frac{\gamma^2}{2} |\Theta|^2$$

then choosing  $K$  so that  $K\gamma^2 = \tilde{\beta} - K$ , we end up with

$$(4.16) \quad \frac{d}{dt} \mathcal{L}_\varrho^2 + \tilde{\beta} \varrho^2 \left( \frac{2}{1 + \gamma^2} |A|^2 + \frac{\gamma^2}{1 + \gamma^2} |\Omega|^2 + 2|\Theta|^2 \right) \leq 0.$$

Using again Young’s inequality, we discover that there exists some constant  $C_0 > 0$  depending only on  $\beta$  and  $\gamma$  so so that

$$(4.17) \quad C_0^{-1} \mathcal{L}_\varrho^2 \leq |(A, \varrho A, \Omega, \Theta)|^2 \leq C_0 \mathcal{L}_\varrho^2.$$

This in particular implies that for all fixed  $\rho_0 > 0$  we have for some constant  $c_0$  depending only on  $\rho_0$ ,  $\beta$  and  $\gamma$ ,

$$\tilde{\beta} \left( \frac{2}{1+\gamma^2} |A|^2 + \frac{\gamma^2}{1+\gamma^2} |\Omega|^2 + 2|\Theta|^2 \right) \geq c_0 \mathcal{L}_\rho^2 \quad \text{for all } 0 \leq \varrho \leq \varrho_0.$$

Therefore, reverting to (4.16),

$$\mathcal{L}_\rho^2(t) \leq e^{-c_0 \varrho^2 t} \mathcal{L}_\rho^2(0),$$

whence using (4.17),

$$(4.18) \quad |(A, \Omega, \Theta)| \leq C e^{-\frac{c_0}{2} t \varrho^2} |(A, \Omega, \Theta)(0)| \quad \text{for all } t \geq 0 \text{ and } 0 \leq \varrho \leq \varrho_0.$$

Multiplying both sides by  $\varphi(2^{-j}\xi)$  with  $|\xi| = \varrho$ , then taking the  $L^2$  norm and using Fourier-Plancherel theorem, we end up for all  $j \leq j_0$  with

$$(4.19) \quad \|(a_{L,j}, \omega_{L,j}, \theta_{L,j})(t)\|_{L^2} \lesssim e^{-\frac{c_0}{2} 2^{2j} t} \|(a_{L,j}, \omega_{L,j}, \theta_{L,j})(0)\|_{L^2}.$$

Now, as the divergence free part  $\mathcal{P}u_L$  of  $u_L$  satisfies the heat equation

$$\partial_t \mathcal{P}u_L - \tilde{\mu} \Delta u_L = 0,$$

we have for all  $j \in \mathbb{Z}$ ,

$$(4.20) \quad \|\mathcal{P}u_{L,j}(t)\|_{L^2} \leq e^{-\tilde{\mu} 2^{2j} t} \|\mathcal{P}u_{L,j}(0)\|_{L^2} \quad \text{with } \tilde{\mu} := \mu/\nu.$$

Then putting (4.19) and (4.20) together completes the proof of the lemma.  $\square$

Set  $U \triangleq (a, v, \theta)$  and  $U_0 \triangleq (a_0, v_0, \theta_0)$ . From Lemma 4.1 and the obvious inequality

$$\sup_{t \geq 0} \sum_{j \in \mathbb{Z}} t^{\frac{s_1+s}{2}} 2^{j(s_1+s)} e^{-c_0 2^{2j} t} \leq C_s < +\infty \quad \text{if } s + s_1 > 0,$$

we get (4.2) (see [14] for more details). Hence we have

$$(4.21) \quad \left\| \int_0^t E(t-\tau)(f, g, k)(\tau) d\tau \right\|_{\dot{B}_{2,1}^{s_1}}^\ell \lesssim \int_0^t \langle t-\tau \rangle^{-\frac{s_1+s}{2}} \|(f, g, k)(\tau)\|_{\dot{B}_{2,\infty}^{-s_1}}^\ell d\tau.$$

We claim that if  $p$  and  $s_1$  fulfill (1.8) and (2.2), respectively, then we have for all  $t \geq 0$ ,

$$(4.22) \quad \int_0^t \langle t-\tau \rangle^{-\frac{s_1+s}{2}} \|(f, g, k)(\tau)\|_{\dot{B}_{2,\infty}^{-s_1}}^\ell d\tau \lesssim \langle t \rangle^{-\frac{s_1+s}{2}} (\mathcal{D}_p^2(t) + \mathcal{X}_p^2(t)).$$

In order to prove our claim, we shall use repeatedly the following obvious inequality that is satisfied whenever  $0 \leq \sigma_1 \leq \sigma_2$  and  $\sigma_2 > 1$ :

$$(4.23) \quad \int_0^t \langle t-\tau \rangle^{-\sigma_1} \langle \tau \rangle^{-\sigma_2} d\tau \lesssim \langle t \rangle^{-\sigma_1} \quad \text{with } \langle t \rangle \triangleq \sqrt{1+t^2}.$$

We shall also take advantage of the embeddings  $L^r \hookrightarrow \dot{B}_{2,\infty}^{-s_1}$  and  $\dot{B}_{p,1}^\sigma \hookrightarrow L^{2r}$  with

$$s_1 \triangleq \frac{d}{r} - \frac{d}{2} \quad \text{and} \quad \sigma \triangleq \frac{d}{p} - \frac{d}{2r}.$$

Let us emphasize that Condition (2.2) is equivalent to  $\frac{p}{2} \leq r \leq \min(2, \frac{d}{2})$ .

All the terms in  $f$ ,  $g$  and  $h$ , but  $I(a)\Delta\theta^h$  will be treated thanks to the following inequalities that follow from Hölder inequality and the above embeddings:

$$(4.24) \quad \|FG\|_{\dot{B}_{2,\infty}^{-s_1}} \lesssim \|FG\|_{L^r} \leq \|F\|_{L^{2r}} \|G\|_{L^{2r}} \lesssim \|F\|_{\dot{B}_{p,1}^\sigma} \|G\|_{\dot{B}_{p,1}^\sigma}.$$

We shall often use the fact that, because  $\sigma + 2 \leq \frac{d}{2} + 1$ ,

$$(4.25) \quad \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{d}{4r} + \frac{k}{2}} \|(\nabla^k a^\ell, \nabla^k v^\ell, \nabla^k \theta^\ell)(\tau)\|_{L^{2r}} \lesssim \mathcal{D}_p(t) \quad \text{for } k = 0, 1, 2.$$

This is just a consequence of

$$\|D^k z\|_{L^{2r}}^\ell \lesssim \|z\|_{\dot{B}_{p,1}^{\sigma+k}}^\ell \lesssim \|z\|_{\dot{B}_{2,1}^{k+\frac{d}{2}-\frac{d}{2r}}}^\ell$$

and of the fact that we have  $-s_1 < k + \frac{d}{2} - \frac{d}{2r} \leq \frac{d}{2} + 1$  for  $k \in \{0, 1, 2\}$ .

Then using also that  $\sigma \leq \frac{d}{p} - 1$  (as  $r \leq \frac{d}{2}$ ), one can write that

$$\|z\|_{\dot{B}_{p,1}^\sigma}^h \lesssim \|z\|_{\dot{B}_{p,1}^{\frac{d}{p}-1}}^h.$$

Therefore as obviously  $\alpha \geq \frac{d}{4r} + \frac{1}{2}$  for small enough  $\varepsilon$ , we have

$$(4.26) \quad \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{d}{4r}} \|(v, a)(\tau)\|_{L^{2r}} + \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{d}{4r} + \frac{1}{2}} \|\nabla a(\tau)\|_{L^{2r}} \lesssim \mathcal{D}_p(t).$$

Combining with (4.24), one can thus bound the terms  $v \cdot \nabla a$  and  $K_1(a)\nabla a$  as follows:

$$\|v \cdot \nabla a\|_{\dot{B}_{2,\infty}^{-s_1}} + \|K_1(a)\nabla a\|_{\dot{B}_{2,\infty}^{-s_1}} \lesssim \|(a, v)\|_{L^{2r}} \|\nabla a\|_{L^{2r}} \lesssim \langle t \rangle^{-\frac{d}{2r} - \frac{1}{2}} \mathcal{D}_p^2(t).$$

Now, as  $\frac{d}{2r} + \frac{1}{2} \geq \frac{1}{2}(s_1 + s)$  for all  $s \leq 1 + \frac{d}{2}$ , we get for all  $t \geq 0$ ,

$$\sup_{-s_1 + \varepsilon \leq s \leq 1 + \frac{d}{2}} \int_0^t \langle t - \tau \rangle^{-\frac{s_1+s}{2}} (\|v \cdot \nabla a\|_{\dot{B}_{2,\infty}^{-s_1}}^\ell + \|K_1(a)\nabla a\|_{\dot{B}_{2,\infty}^{-s_1}}^\ell) d\tau \lesssim \langle t \rangle^{-\frac{s_1+s}{2}} (\mathcal{D}_p^2(t) + \mathcal{X}_p^2(t)).$$

The terms  $a \operatorname{div} v^\ell$ ,  $v \cdot \nabla v^\ell$ ,  $I(a)\Delta v^\ell$ ,  $K_2(a)\nabla \theta^\ell$ ,  $\theta^\ell \nabla K_3(a)$ ,  $v \cdot \nabla \theta^\ell$ ,  $I(a)\Delta \theta^\ell$ ,  $\tilde{K}_1(a) \operatorname{div} v^\ell$  and  $\tilde{K}_2(a)\theta^\ell \operatorname{div} v^\ell$  may be handled along the same lines.

To deal with the other terms in  $f$ ,  $g$  and  $k$ , we shall also often use the fact that

$$(4.27) \quad \|t^{\frac{s_1}{2} + \frac{d}{4} - \frac{\varepsilon}{2}}(a, v, \theta)\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}})} \lesssim \mathcal{D}_p(t).$$

This may be proved by decomposing functions  $a$ ,  $v$  and  $\theta$  in low and high frequencies, using the definition of  $\mathcal{D}_p(t)$  and the fact that, because  $p \geq 2$  and  $\alpha \geq \frac{s_1}{2} + \frac{d}{4} - \frac{\varepsilon}{2}$ ,

$$\begin{aligned} \|t^{\frac{s_1}{2} + \frac{d}{4} - \frac{\varepsilon}{2}}(a, v, \theta)\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}})} &\lesssim \|t^{\frac{s_1}{2} + \frac{d}{4} - \frac{\varepsilon}{2}}(a, v, \theta)\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \|t^{\frac{s_1}{2} + \frac{d}{4} - \frac{\varepsilon}{2}}(a, v, \theta)\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}})}^h \\ &\lesssim \|t^{\frac{s_1}{2} + \frac{d}{4} - \frac{\varepsilon}{2}}(a, v, \theta)\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-\varepsilon})}^\ell + \|t^\alpha(a, v, \theta)\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}})}^h. \end{aligned}$$

As an example, let us bound  $K_2(a)\nabla \theta^h$ . Then we use that, owing to (4.24), if  $t \geq 2$ ,

$$\begin{aligned} &\int_0^t \langle t - \tau \rangle^{-\frac{s_1+s}{2}} \|(K_2(a)\nabla \theta^h)(\tau)\|_{\dot{B}_{2,\infty}^{-s_1}} d\tau \\ &\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s_1+s}{2}} \|a(\tau)\|_{L^{2r}} \|\nabla \theta^h(\tau)\|_{L^{2r}} d\tau = \left( \int_0^1 + \int_1^t \right) (\cdots) d\tau \triangleq I_1 + I_2. \end{aligned}$$

Using the fact that

$$\|\nabla \theta^h\|_{L^{2r}} \lesssim \|\theta\|_{\dot{B}_{p,1}^{\sigma+1}}^h \lesssim \|\theta\|_{\dot{B}_{p,1}^{\frac{d}{p}}}^h,$$

and remembering the definitions of  $\mathcal{X}_p(t)$  and  $\mathcal{D}_p(t)$ , and Inequality (4.26), we obtain

$$\begin{aligned} I_1 &\lesssim \langle t \rangle^{-\frac{s_1+s}{2}} \left( \sup_{0 \leq \tau \leq 1} \|a(\tau)\|_{L^{2r}} \right) \int_0^1 \|\theta(\tau)\|_{\dot{B}_{p,1}^{\frac{d}{p}}}^h d\tau \\ (4.28) \quad &\lesssim \langle t \rangle^{-\frac{s_1+s}{2}} \mathcal{D}_p(1) \mathcal{X}_p(1). \end{aligned}$$

Next, because  $\langle \tau \rangle \approx \tau$  when  $\tau \geq 1$ , one has for all  $-s_1 < s \leq \frac{d}{2} + 1$ ,

$$\begin{aligned} I_2 &= \int_1^t \langle t - \tau \rangle^{-\frac{s_1+s}{2}} \langle \tau \rangle^{-\frac{d}{2r} - \frac{1}{2}} \left( \langle \tau \rangle^{\frac{d}{4r}} \|a(\tau)\|_{L^{2r}} \right) \left( \tau^{\frac{d}{4r} + \frac{1}{2}} \|\nabla \theta^h(\tau)\|_{L^{2r}} \right) d\tau \\ &\lesssim \sup_{1 \leq \tau \leq t} \left( \langle \tau \rangle^{\frac{d}{4r}} \|a(\tau)\|_{L^{2r}} \right) \sup_{1 \leq \tau \leq t} \left( \tau^\alpha \|\theta(\tau)\|_{\dot{B}_{p,1}^{\frac{d}{p}}}^h \right) \int_1^t \langle t - \tau \rangle^{-\frac{s_1+s}{2}} \langle \tau \rangle^{-\frac{d}{2r} - \frac{1}{2}} d\tau \\ &\lesssim \langle t \rangle^{-\frac{s_1+s}{2}} \mathcal{D}_p^2(t). \end{aligned}$$

Therefore, for  $t \geq 2$ , we arrive at

$$(4.29) \quad \int_0^t \langle t - \tau \rangle^{-\frac{s_1+s}{2}} \|(K_2(a) \nabla \theta^h)(\tau)\|_{\dot{B}_{2,\infty}^{-s_1}} d\tau \lesssim \langle t \rangle^{-\frac{s_1+s}{2}} \left( \mathcal{D}_p(t) \mathcal{X}_p(t) + \mathcal{D}_p^2(t) \right).$$

The case  $t \leq 2$  is obvious as  $\langle t \rangle \approx 1$  and  $\langle t - \tau \rangle \approx 1$  for  $0 \leq \tau \leq t \leq 2$ , and

$$(4.30) \quad \int_0^t \|K_2(a) \nabla \theta^h\|_{L^r} d\tau \leq \|a\|_{L_t^\infty(L^{2r})} \|\nabla \theta^h\|_{L_t^1(L^{2r})} \lesssim \mathcal{D}_p(t) \mathcal{X}_p(t).$$

The terms with  $a \operatorname{div} v^h$ ,  $I(a) \Delta v^h$ ,  $v \cdot \nabla v^h$ ,  $\theta^h \nabla K_3(a)$ ,  $v \cdot \nabla \theta^h$  and  $\tilde{K}_1(a) \operatorname{div} v^h$  may be treated along the same lines. For the term corresponding to  $\frac{Q(\nabla v, \nabla v)}{1+a}$  one may write, thanks to (4.1),

$$\begin{aligned} \int_0^t \langle t - \tau \rangle^{-\frac{s_1+s}{2}} \left\| \frac{Q(\nabla v, \nabla v)}{1+a} \right\|_{\dot{B}_{2,\infty}^{-s_1}}^\ell d\tau &\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s_1+s}{2}} \|\nabla v(\tau)\|_{L^{2r}}^2 d\tau \\ &\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s_1+s}{2}} (\|\nabla v^\ell(\tau)\|_{L^{2r}}^2 + \|\nabla v^h(\tau)\|_{L^{2r}}^2) d\tau. \end{aligned}$$

Now, (4.25) implies that

$$\begin{aligned} (4.31) \quad &\int_0^t \langle t - \tau \rangle^{-\frac{s_1+s}{2}} \|\nabla v^\ell(\tau)\|_{L^{2r}}^2 d\tau \\ &\lesssim \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{d}{4r} + \frac{1}{2}} \|\nabla v^\ell(\tau)\|_{L^{2r}} \right)^2 \int_0^t \langle t - \tau \rangle^{-\frac{s_1+s}{2}} \langle \tau \rangle^{-\frac{d}{2r} - 1} d\tau \lesssim \langle t \rangle^{-\frac{s_1+s}{2}} \mathcal{D}_p^2(t) \end{aligned}$$

and we have if  $t \geq 2$ ,

$$\int_0^t \langle t - \tau \rangle^{-\frac{s_1+s}{2}} \|\nabla v^h(\tau)\|_{L^{2r}}^2 d\tau = \left( \int_0^1 + \int_1^t \right) (\cdots) d\tau \triangleq J_1 + J_2.$$

Using that  $\sigma \leq \frac{d}{p} - 1$  and interpolation (see Lemma 3.2), we arrive at

$$\|\nabla v^h\|_{L^{2r}} \lesssim \|\nabla v^h\|_{\dot{B}_{p,1}^\sigma} \lesssim \|\nabla v^h\|_{\dot{B}_{p,1}^{\frac{d}{p}-2}}^{\frac{1}{2}} \|\nabla v^h\|_{\dot{B}_{p,1}^{\frac{d}{p}}}^{\frac{1}{2}},$$

which leads to

$$(4.32) \quad J_1 \lesssim \langle t \rangle^{-\frac{s_1+s}{2}} \int_0^1 \|\nabla v^h(\tau)\|_{L^{2r}}^2 d\tau \lesssim \langle t \rangle^{-\frac{s_1+s}{2}} \mathcal{X}_p^2(1).$$

For  $J_2$ , we get for all  $-s_1 < s \leq \frac{d}{2} + 1$ ,

$$J_2 \lesssim \int_1^t \langle t - \tau \rangle^{-\frac{s_1+s}{2}} \langle \tau \rangle^{-\frac{d}{2r} - \frac{1}{2}} \left( \|\tau^{\frac{d}{4r} + \frac{1}{4}} \nabla v^h\|_{\dot{B}_{p,1}^{\frac{d}{2r} - 1}} \right)^2 d\tau \lesssim \langle t \rangle^{-\frac{s_1+s}{2}} \mathcal{D}_p^2(t).$$

So finally, if  $t \geq 2$ ,

$$(4.33) \quad \int_0^t \langle t - \tau \rangle^{-\frac{s_1+s}{2}} \|\nabla v^h(\tau)\|_{L^{2r}}^2 d\tau \lesssim \langle t \rangle^{-\frac{s_1+s}{2}} \left( \mathcal{X}_p^2(t) + \mathcal{D}_p^2(t) \right).$$

The above inequality also holds true for  $t \leq 2$  since  $\langle t \rangle \approx 1$  and  $\langle t - \tau \rangle \approx 1$  for  $0 \leq \tau \leq t \leq 2$ . Therefore, combining with (4.31) and (4.33), we end up with

$$(4.34) \quad \int_0^t \langle t - \tau \rangle^{-\frac{s_1+s}{2}} \left\| \frac{Q(\nabla v, \nabla v)}{1+a} \right\|_{\dot{B}_{2,\infty}^{-s_1}}^\ell d\tau \lesssim \langle t \rangle^{-\frac{s_1+s}{2}} \left( \mathcal{X}_p^2(t) + \mathcal{D}_p^2(t) \right).$$

The term  $\tilde{K}_2(a)\theta^h \operatorname{div} v^\ell$  may be treated as  $K_2(a)\nabla \theta^h$  and  $\tilde{K}_2(a)\theta^\ell \operatorname{div} v^h$ , as for example  $a \operatorname{div} v^h$ . To bound  $\tilde{K}_2(a)\theta^h \operatorname{div} v^h$ , one has to proceed slightly differently. If  $t \geq 2$  then we start as usual with

$$\int_0^t \langle t - \tau \rangle^{-\frac{s_1+s}{2}} \|\theta^h(\tau)\|_{L^{2r}} \|\operatorname{div} v^h(\tau)\|_{L^{2r}} d\tau = \left( \int_0^1 + \int_1^t \right) (\cdot \cdot \cdot) d\tau \triangleq \tilde{J}_1 + \tilde{J}_2.$$

For  $J_1$ , one can write

$$\begin{aligned} \tilde{J}_1 &\lesssim \langle t \rangle^{-\frac{s_1+s}{2}} \|\theta^h\|_{L^2([0,1], L^{2r})} \|\operatorname{div} v^h\|_{L^2([0,1], L^{2r})} \\ &\lesssim \langle t \rangle^{-\frac{s_1+s}{2}} \left( \|\theta\|_{L^1([0,1], \dot{B}_{p,1}^{\frac{d}{p}})}^h + \|\theta\|_{\tilde{L}^\infty([0,1], \dot{B}_{p,1}^{\frac{d}{p}-2})}^h \right) \left( \|v\|_{L^1([0,1], \dot{B}_{p,1}^{\frac{d}{p}+1})}^h + \|v\|_{\tilde{L}^\infty([0,1], \dot{B}_{p,1}^{\frac{d}{p}-1})}^h \right) \\ &\lesssim \langle t \rangle^{-\frac{s_1+s}{2}} \mathcal{X}_p^2(1) \end{aligned}$$

and for all  $-s_1 < s \leq \frac{d}{2} + 1$ , we have, thanks to (4.25),

$$\begin{aligned} \tilde{J}_2 &\lesssim \int_1^t \langle t - \tau \rangle^{-\frac{s_1+s}{2}} \langle \tau \rangle^{-\frac{d}{2r} - \frac{1}{2}} \left( \tau^{\frac{d}{4r}} \|\theta^h(\tau)\|_{L^{2r}} \right) \left( \tau^{\frac{d}{4r} + \frac{1}{2}} \|\operatorname{div} v^h(\tau)\|_{L^{2r}} \right) d\tau \\ &\lesssim \langle t \rangle^{-\frac{s_1+s}{2}} \mathcal{D}_p^2(t). \end{aligned}$$

The case  $t \leq 2$  is left to the reader.

To bound the term corresponding to  $I(a)\Delta \theta^h$ , one has to consider the cases  $2 \leq p \leq \frac{d}{2}$  and  $p > \frac{d}{2}$  separately. If  $2 \leq p \leq \frac{d}{2}$  then we have, denoting  $\frac{1}{q} \triangleq \frac{1}{p} + \frac{1}{2r}$  and  $s_2 \triangleq \frac{d}{p} + \frac{d}{2r} - \frac{d}{2}$  and using the embedding  $L^q \hookrightarrow \dot{B}_{2,\infty}^{-s_2}$ ,

$$(4.35) \quad \|I(a)\Delta \theta^h\|_{\dot{B}_{2,\infty}^{-s_2}}^\ell \lesssim \|I(a)\Delta \theta^h\|_{L^q}^\ell \lesssim \|a\|_{L^{2r}} \|\Delta \theta^h\|_{L^p} \lesssim \|a\|_{L^{2r}} \|\theta\|_{\dot{B}_{p,1}^{\frac{d}{p}}}^h$$

By repeating the procedure leading to (4.29)-(4.30) and using that  $-s_2 \leq -s_1$ , we get

$$\begin{aligned} \int_0^t \langle t - \tau \rangle^{-\frac{s_1+s}{2}} \|I(a)\Delta \theta^h\|_{\dot{B}_{2,\infty}^{-s_1}}^\ell d\tau &\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s_1+s}{2}} \|I(a)\Delta \theta^h\|_{\dot{B}_{2,\infty}^{-s_2}}^\ell d\tau \\ &\lesssim \langle t \rangle^{-\frac{s_1+s}{2}} \left( \mathcal{D}_p^2(t) + \mathcal{X}_p(t) \mathcal{D}_p(t) \right). \end{aligned}$$



In the case  $d/2 < p < d$ , the regularity exponent  $\frac{d}{p} - 2$  is negative, which precludes  $\Delta\theta^h$  to be in any Lebesgue space. However, it follows from (3.15), taking  $\sigma = 2 - d/p$ , that

$$\|I(a)\Delta\theta^h\|_{\dot{B}_{2,\infty}^{-s_0}}^\ell \leq C(\|I(a)\|_{L^{p^*}} + \|I(a)\|_{\dot{B}_{p,1}^{2-\frac{d}{p}}})\|\Delta\theta^h\|_{\dot{B}_{p,\infty}^{\frac{d}{p}-2}}$$

with  $s_0 \triangleq \frac{2d}{p} - \frac{d}{2}$  and  $\frac{1}{p^*} = \frac{1}{2} - \frac{1}{p}$ .

Now, Proposition 3.5 and obvious embedding ensure that

$$\|I(a)\|_{\dot{B}_{p,1}^{2-\frac{d}{p}}} \lesssim \|a\|_{\dot{B}_{p,1}^{2-\frac{d}{p}}} \lesssim \|a\|_{\dot{B}_{2,1}^{2-s_0}}^\ell + \|a\|_{\dot{B}_{p,1}^{\frac{d}{p}}}^h$$

and, because  $p^* \geq p$ , we have  $\dot{B}_{2,1}^{\frac{d}{p}} \hookrightarrow L^{p^*}$  and  $\dot{B}_{p,1}^{s_0} \hookrightarrow L^{p^*}$ , whence

$$\|I(a)\|_{L^{p^*}} \lesssim \|a\|_{L^{p^*}} \lesssim \|a\|_{\dot{B}_{2,1}^{\frac{d}{p}}}^\ell + \|a\|_{\dot{B}_{p,1}^{\frac{d}{p}}}^h.$$

Therefore we eventually get

$$(4.36) \quad \|I(a)\Delta\theta^h\|_{\dot{B}_{2,\infty}^{-s_0}}^\ell \lesssim (\|a\|_{\dot{B}_{2,1}^{2-s_0}}^\ell + \|a\|_{\dot{B}_{p,1}^{\frac{d}{p}}}^\ell + \|a\|_{\dot{B}_{p,1}^{\frac{d}{p}}}^h)\|\theta^h\|_{\dot{B}_{p,1}^{\frac{d}{p}}}.$$

If  $t \geq 2$  then (4.36) implies that

$$(4.37) \quad \begin{aligned} \int_0^t \langle t-\tau \rangle^{-\frac{s_1+s}{2}} \|I(a)\Delta\theta^h\|_{\dot{B}_{2,\infty}^{-s_0}}^\ell d\tau &\lesssim \int_0^t \langle t-\tau \rangle^{-\frac{s_1+s}{2}} (\|a\|_{\dot{B}_{2,1}^{\min(2-s_0, \frac{d}{p})}}^\ell + \|a\|_{\dot{B}_{p,1}^{\frac{d}{p}}}^h) \|\theta^h\|_{\dot{B}_{p,1}^{\frac{d}{p}}}^\ell d\tau \\ &= \left( \int_0^1 + \int_1^t \right) (\cdots) d\tau \triangleq \tilde{I}_1 + \tilde{I}_2. \end{aligned}$$

On the one hand, the definitions of  $\mathcal{D}_p$  and  $\mathcal{X}_p$  ensure that

$$(4.38) \quad \tilde{I}_1 \lesssim \langle t \rangle^{-\frac{s_1+s}{2}} \mathcal{D}_p(1) \mathcal{X}_p(1).$$

On the other hand, thanks to the fact that

$$(4.39) \quad \sup_{\tau \in [1, t]} \tau^\alpha \|\theta^h(\tau)\|_{\dot{B}_{p,1}^{\frac{d}{p}}} \lesssim \mathcal{D}_p(t),$$

that  $\alpha \geq \frac{s_1+s}{2}$  for all  $s \leq 1 + \frac{d}{2}$  and that  $2 - s_0 > -s_1$  for all  $s_1$  satisfying (2.2) (if  $p > \frac{d}{2}$ ), we end up with

$$\begin{aligned} \tilde{I}_2 &\lesssim \int_1^t \langle t-\tau \rangle^{-\frac{s_1+s}{2}} \langle \tau \rangle^{-\alpha} (\|a\|_{\dot{B}_{2,1}^{\min(2-s_0, \frac{d}{p})}}^\ell + \|a\|_{\dot{B}_{p,1}^{\frac{d}{p}}}^h) (\tau^\alpha \|\theta^h\|_{\dot{B}_{p,1}^{\frac{d}{p}}}^\ell) d\tau \\ &\lesssim \langle t \rangle^{-\frac{s_1+s}{2}} \left( \mathcal{D}_p^2(t) + \mathcal{X}_p(t) \mathcal{D}_p(t) \right). \end{aligned}$$

Putting together with (4.37) and using that  $-s_0 \leq -s_1$ , we thus get if  $t \geq 2$ ,

$$\int_0^t \langle t-\tau \rangle^{-\frac{s_1+s}{2}} \|I(a)\Delta\theta^h\|_{\dot{B}_{2,\infty}^{-s_1}}^\ell d\tau \lesssim \langle t \rangle^{-\frac{s_1+s}{2}} \left( \mathcal{D}_p^2(t) + \mathcal{X}_p(t) \mathcal{D}_p(t) \right).$$

That the above inequality holds for  $t \leq 2$  is a direct consequence of the definitions of  $\mathcal{D}_p$  and  $\mathcal{X}_p$ .

Putting together all the above estimates completes the proof of Inequality (4.22). Then, combining with (4.2) for bounding the term of (4.4) pertaining to the data, we get

$$(4.40) \quad \langle t \rangle^{\frac{s_1+s}{2}} \|(a, v, \theta)(t)\|_{\dot{B}_{2,1}^s}^\ell \lesssim \mathcal{D}_{p,0} + \mathcal{D}_p^2(t) + \mathcal{X}_p^2(t) \quad \text{for all } t \geq 0 \text{ and } -s_1 < s \leq \frac{d}{2} + 1.$$

*Step 2: Decay estimates for the high frequencies of  $(\nabla a, v, \theta)$ .* Let us first recall the following elementary result (see the proof in [14]).

**Lemma 4.2.** *Let  $X : [0, T] \rightarrow \mathbb{R}_+$  be a continuous function. Assume that  $X^p$  is differentiable for some  $p \geq 1$  and satisfies*

$$\frac{1}{p} \frac{d}{dt} X^p + M X^p \leq F X^{p-1}$$

*for some constant  $M \geq 0$  and measurable function  $F : [0, T] \rightarrow \mathbb{R}_+$ .*

*Denote  $X_\varepsilon = (X^p + \varepsilon^p)^{1/p}$  for  $\varepsilon > 0$ . Then it holds that*

$$\frac{d}{dt} X_\varepsilon + M X_\varepsilon \leq F + M \varepsilon.$$

Let  $\mathcal{P} \triangleq \text{Id} + \nabla(-\Delta)^{-1} \text{div}$  be the Leray projector onto divergence-free vector fields. It follows from (2.1) that  $\mathcal{P}u$  satisfies the following ordinary heat equation:

$$\partial_t \mathcal{P}v - \tilde{\mu} \Delta \mathcal{P}v = \mathcal{P}g.$$

Applying  $\dot{\Delta}_j$  to the above equation gives for all  $j \in \mathbb{Z}$ ,

$$\partial_t \mathcal{P}v_j - \tilde{\mu} \Delta \mathcal{P}v_j = \mathcal{P}g_j \quad \text{with} \quad v_j \triangleq \dot{\Delta}_j v \quad \text{and} \quad g_j \triangleq \dot{\Delta}_j g.$$

According to Proposition 3.2, we thus end up for some constant  $c_p > 0$  with

$$\frac{1}{p} \frac{d}{dt} \|\mathcal{P}v_j\|_{L^p}^p + c_p \tilde{\mu} 2^{2j} \|\mathcal{P}v_j\|_{L^p}^p \leq \|\mathcal{P}g_j\|_{L^p} \|\mathcal{P}v_j\|_{L^p}^{p-1}.$$

Hence, using the notation  $\|\cdot\|_{\varepsilon, L^p} \triangleq (\|\cdot\|_{L^p}^p + \varepsilon^p)^{1/p}$ , it follows from Lemma 4.2 that for all  $\varepsilon > 0$ ,

$$(4.41) \quad \frac{d}{dt} \|\mathcal{P}v_j\|_{\varepsilon, L^p} + c_p \tilde{\mu} 2^{2j} \|\mathcal{P}v_j\|_{\varepsilon, L^p} \leq \|\mathcal{P}g_j\|_{L^p} + c_p \tilde{\mu} 2^{2j} \varepsilon.$$

Next, we observe that  $(a, w)$  with  $w$  being the *effective velocity* defined in (4.3) fulfills

$$\begin{cases} \partial_t w - \Delta w = \nabla(-\Delta)^{-1}(f - \text{div} g) - \gamma \nabla \theta + w - (-\Delta)^{-1} \nabla a, \\ \partial_t a + a = f - \text{div} w. \end{cases}$$

Arguing as for  $\mathcal{P}v$ , one can arrive at

$$(4.42) \quad \frac{d}{dt} \|w_j\|_{\varepsilon, L^p} + c_p 2^{2j} \|w_j\|_{\varepsilon, L^p} \leq \|\nabla(-\Delta)^{-1}(f_j - \text{div} g_j)\|_{L^p} + \|\gamma \nabla \theta_j + w_j - (-\Delta)^{-1} \nabla a_j\|_{L^p} + c_p 2^{2j} \varepsilon,$$

and, denoting  $R_j^i \triangleq [u \cdot \nabla, \partial_i \dot{\Delta}_j] a$  for  $i = 1, \dots, d$ ,

$$(4.43) \quad \frac{d}{dt} \|\nabla a_j\|_{\varepsilon, L^p} + \|\nabla a_j\|_{\varepsilon, L^p} \leq \left( \frac{1}{p} \|\text{div} v\|_{L^\infty} \|\nabla a_j\|_{L^p} + \|\nabla \dot{\Delta}_j(a \text{div} v)\|_{L^p} + C 2^{2j} \|w_j\|_{L^p} + \|R_j\|_{L^p} \right) + \varepsilon.$$

Similarly, as the temperature  $\theta$  satisfies

$$(4.44) \quad \partial_t \theta - \beta \Delta \theta + \gamma \text{div} w + a = k,$$

we have, according to Proposition 3.2, that

$$(4.45) \quad \frac{d}{dt} \|\theta_j\|_{\varepsilon, L^p} + c_p \beta 2^{2j} \|\theta_j\|_{\varepsilon, L^p} \leq \|k_j - \gamma \text{div} w_j - a_j\|_{L^p} + c_p 2^{2j} \varepsilon.$$

Adding up (4.41), (4.42),  $Ac_p \times (4.43)$  and  $B2^{-j} \times (4.45)$  for some  $A, B > 0$  (to be chosen afterward) gives that

$$\begin{aligned} \frac{d}{dt} (\|\mathcal{P}v_j\|_{\varepsilon, L^p} + \|w_j\|_{\varepsilon, L^p} + Ac_p \|\nabla a_j\|_{\varepsilon, L^p} + 2^{-j} \beta B \|\theta_j\|_{\varepsilon, L^p}) + c_p 2^{2j} (\tilde{\mu} \|\mathcal{P}v_j\|_{\varepsilon, L^p} + \|w_j\|_{\varepsilon, L^p}) \\ + Ac_p \|\nabla a_j\|_{\varepsilon, L^p} + c_p \beta B 2^j \|\theta_j\|_{\varepsilon, L^p} \leq (\|\mathcal{P}g_j\|_{L^p} + \|\nabla(-\Delta)^{-1}(f_j - \operatorname{div} g_j)\|_{L^p}) \\ + Ac_p \left( \frac{1}{p} \|\operatorname{div} v\|_{L^\infty} \|\nabla a_j\|_{L^p} + \|\nabla \dot{\Delta}_j(a \operatorname{div} v)\|_{L^p} + \|R_j\|_{L^p} \right) + 2^{-j} B \|k_j\|_{L^p} \\ + 2^{-j} B \|\gamma \operatorname{div} w_j + a_j\|_{L^p} + \|\gamma \nabla \theta_j + w_j - (-\Delta)^{-1} \nabla a_j\|_{L^p} + C A c_p 2^{2j} \|w_j\|_{L^p} + M_\varepsilon, \end{aligned}$$

where  $M_\varepsilon \triangleq (c_p \tilde{\mu} 2^{2j} + c_p 2^{2j} + Ac_p + c_p \beta B 2^j) \varepsilon$ .

Because  $(-\Delta)^{-1}$  is a homogeneous Fourier multiplier of degree  $-2$ , we have

$$(4.46) \quad \|(-\Delta)^{-1} \nabla a_j\|_{L^p} \lesssim 2^{-2j} \|\nabla a_j\|_{L^p} \lesssim 2^{-2j_0} \|\nabla a_j\|_{\varepsilon, L^p} \quad \text{for } j \geq j_0 - 1.$$

Choosing first  $B$  large enough, then  $A$  suitably small, and finally  $j_0$  suitably large, one can absorb the last line of the above inequality by the l.h.s. Hence, there exists a constant  $c_0 > 0$  such that for all  $j \geq j_0 - 1$ ,

$$\begin{aligned} \frac{d}{dt} (\|\mathcal{P}v_j\|_{\varepsilon, L^p} + \|w_j\|_{\varepsilon, L^p} + Ac_p \|\nabla a_j\|_{\varepsilon, L^p} + 2^{-j} \beta B \|\theta_j\|_{\varepsilon, L^p}) + c_0 (\|\mathcal{P}v_j\|_{\varepsilon, L^p} + \|w_j\|_{\varepsilon, L^p}) \\ + Ac_p \|\nabla a_j\|_{\varepsilon, L^p} + 2^{-j} \beta B \|\theta_j\|_{\varepsilon, L^p} \leq (\|g_j\|_{L^p} + \|\dot{\Delta}_j(av)\|_{L^p}) \\ + Ac_p \left( \frac{1}{p} \|\operatorname{div} u\|_{L^\infty} \|\nabla a_j\|_{L^p} + \|\nabla \dot{\Delta}_j(a \operatorname{div} u)\|_{L^p} + \|R_j\|_{L^p} \right) + 2^{-j} B \|k_j\|_{L^p} + M_\varepsilon. \end{aligned}$$

Then, integrating in time and having  $\varepsilon$  tend to 0, we arrive for all  $j \geq j_0 - 1$  at

$$(4.47) \quad e^{c_0 t} \|(\mathcal{P}v_j, w_j, \nabla a_j, 2^{-j} \theta_j)(t)\|_{L^p} \lesssim \|(\mathcal{P}v_j, w_j, \nabla a_j, 2^{-j} \theta_j)(0)\|_{L^p} + \int_0^t e^{c_0 \tau} S_j(\tau) d\tau$$

with  $S_j \triangleq S_j^1 + \dots + S_j^6$  and

$$\begin{aligned} S_j^1 &\triangleq \|\dot{\Delta}_j(av)\|_{L^p}, \quad S_j^2 \triangleq \|g_j\|_{L^p}, \quad S_j^3 \triangleq 2^{-j} \|k_j\|_{L^p}, \\ S_j^4 &\triangleq \|\nabla \dot{\Delta}_j(a \operatorname{div} v)\|_{L^p}, \quad S_j^5 \triangleq \|R_j\|_{L^p}, \quad S_j^6 \triangleq \|\operatorname{div} v\|_{L^\infty} \|\nabla a_j\|_{L^p}. \end{aligned}$$

It is clear that  $(v_j, \nabla a_j, 2^{-j} \theta_j)$  satisfies a similar inequality. Indeed, we have

$$v = w - \nabla(-\Delta)^{-1} a + \mathcal{P}v$$

which leads for  $j \geq j_0 - 1$  to

$$\|v_j\|_{L^p} \lesssim \|w_j\|_{L^p} + \|\mathcal{P}v_j\|_{L^p} + 2^{-2j_0} \|\nabla a_j\|_{L^p}.$$

Therefore, there exists a constant  $c_0 > 0$  such that for all  $j \geq j_0 - 1$  and  $t \geq 0$ ,

$$\|(2^j a_j, v_j, 2^{-j} \theta_j)(t)\|_{L^p} \lesssim e^{-c_0 t} \|(2^j a_j(0), v_j(0), 2^{-j} \theta_j(0))\|_{L^p} + \int_0^t e^{-c_0(t-\tau)} S_j(\tau) d\tau.$$

Now, multiplying both sides by  $\langle t \rangle^\alpha 2^{j(\frac{d}{p}-1)}$ , taking the supremum on  $[0, T]$ , and summing up over  $j \geq j_0 - 1$  yields

$$\begin{aligned} (4.48) \quad & \|\langle t \rangle^\alpha a\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}})}^h + \|\langle t \rangle^\alpha v\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})}^h + \|\langle t \rangle^\alpha \theta\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h \\ & \lesssim \|a_0\|_{\dot{B}_{p,1}^{\frac{d}{p}}}^h + \|v_0\|_{\dot{B}_{p,1}^{\frac{d}{p}-1}}^h + \|\theta_0\|_{\dot{B}_{p,1}^{\frac{d}{p}-2}}^h + \sum_{j \geq j_0-1} \sup_{0 \leq t \leq T} \left( \langle t \rangle^\alpha \int_0^t e^{-c_0(t-\tau)} 2^{j(\frac{d}{p}-1)} S_j(\tau) d\tau \right). \end{aligned}$$

To treat the case  $T \leq 2$ , we use the fact that

$$(4.49) \quad \sum_{j \geq j_0-1} \sup_{0 \leq t \leq 2} \left( \langle t \rangle^\alpha \int_0^t e^{-c_0(t-\tau)} 2^{j(\frac{d}{p}-1)} S_j(\tau) d\tau \right) \lesssim \int_0^2 \sum_{j \geq j_0-1} 2^{j(\frac{d}{p}-1)} S_j(\tau) d\tau.$$

The terms in  $S_j^1, S_j^4, S_j^5$  and  $S_j^6$  as well as those in  $S_j^2$  corresponding to  $v \cdot \nabla v, I(a)\Delta v$  or  $K_1(a)\nabla a$  may be estimated exactly as in [14]. Therefore, it is only a matter of handling the ‘new’ terms in  $S_j^2$ , and  $S_j^3$ , that is

$$(4.50) \quad K_2(a)\nabla\theta, \theta\nabla K_3(a), v \cdot \nabla\theta, I(a)\Delta\theta, \frac{Q(\nabla v, \nabla v)}{1+a}, \tilde{K}_1(a)\operatorname{div} v \text{ and } \tilde{K}_2(a)\theta\operatorname{div} v.$$

To this end, we shall often use the fact that, by interpolation, we have

$$(4.51) \quad \|a\|_{L_t^2(\dot{B}_{p,1}^{\frac{d}{p}})} + \|v\|_{L_t^2(\dot{B}_{p,1}^{\frac{d}{p}})} \lesssim \mathcal{X}_p(t).$$

For  $K_2(a)\nabla\theta$ , we still use the decomposition

$$K_2(a)\nabla\theta = K_2(a)\nabla\theta^\ell + K_2(a)\nabla\theta^h.$$

Thanks to Hölder inequality and Propositions 3.3 and 3.5, we get

$$(4.52) \quad \|K_2(a)\nabla\theta\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}-1})}^h \lesssim \|a\|_{L_t^2(\dot{B}_{p,1}^{\frac{d}{p}})} \|\nabla\theta^\ell\|_{L_t^2(\dot{B}_{p,1}^{\frac{d}{p}-1})} + \|a\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}})} \|\nabla\theta^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}-1})}.$$

Now, interpolation and embedding (recall that  $p \geq 2$ ) imply that

$$\|\nabla\theta^\ell\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}-1})} \lesssim \|\theta^\ell\|_{L_T^1(\dot{B}_{p,1}^{\frac{d}{p}-1})}^{\frac{1}{2}} \|\theta^\ell\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}+1})}^{\frac{1}{2}} \lesssim \|\theta\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})}^\ell + \|\theta\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell \lesssim \mathcal{X}_p(t)$$

and

$$\|a\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}})} \lesssim \|a\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})}^\ell + \|a\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}})}^h \lesssim \mathcal{X}_p(t).$$

Hence, we arrive at

$$(4.53) \quad \|K_2(a)\nabla\theta\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}-1})}^h \lesssim \mathcal{X}_p^2(t).$$

Similarly, it follows from Proposition 3.3 that

$$\begin{aligned} \|\theta\nabla K_3(a)\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}-1})}^h &\lesssim \|\theta^\ell\|_{L_t^2(\dot{B}_{p,1}^{\frac{d}{p}})} \|\nabla K_3(a)\|_{L_T^2(\dot{B}_{p,1}^{\frac{d}{p}-1})} + \|\theta^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}})} \|\nabla K_3(a)\|_{L_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})} \\ &\lesssim \mathcal{X}_p^2(t). \end{aligned}$$

Next, for  $v \cdot \nabla\theta^\ell$ , we use the following obvious inequality

$$(4.54) \quad \|z\|_{\tilde{L}_T^\infty(\dot{B}_{q,1}^{\frac{d}{q}-\varsigma})}^h \lesssim 2^{-j_0\varsigma} \|z\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}})}^h \quad \text{for } q = 2, p \text{ and } \varsigma \geq 0,$$

which, combined with Proposition 3.3 implies that

$$\begin{aligned} \|v \cdot \nabla\theta^\ell\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h &\lesssim 2^{-j_0} \|v \cdot \nabla\theta^\ell\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}-1})} \\ &\lesssim \|v\|_{L_t^2(\dot{B}_{p,1}^{\frac{d}{p}})} \|\nabla\theta^\ell\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}-1})}, \end{aligned}$$

and for  $v \cdot \nabla\theta^h$ , we have

$$\|v \cdot \nabla\theta^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h \lesssim \|v\|_{L_t^2(\dot{B}_{p,1}^{\frac{d}{p}})} \|\nabla\theta^h\|_{L_t^2(\dot{B}_{p,1}^{\frac{d}{p}-2})}.$$

Therefore we have

$$(4.55) \quad \|v \cdot \nabla \theta\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h \lesssim \mathcal{X}_p^2(t).$$

Similarly,

$$\|I(a)\Delta\theta\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h \lesssim \|a\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}})} \left( 2^{-j_0} \|\Delta\theta^\ell\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|\Delta\theta^h\|_{L_T^1(\dot{B}_{p,1}^{\frac{d}{p}-2})} \right) \lesssim \mathcal{X}_p^2(t).$$

Because  $p < d$  and  $d \geq 3$ , it follows from Proposition 3.3 and (4.1) that

$$(4.56) \quad \left\| \frac{Q(\nabla v, \nabla v)}{1+a} \right\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h \lesssim (1 + \|a\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}})}) \|\nabla v\|_{L_t^2(\dot{B}_{p,1}^{\frac{d}{p}-1})}^2 \lesssim \mathcal{X}_p^2(t)$$

and

$$(4.57) \quad \|\tilde{K}_1(a) \operatorname{div} v\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h \lesssim 2^{-j_0} \|a\|_{L_T^2(\dot{B}_{p,1}^{\frac{d}{p}})} \|\operatorname{div} v\|_{L_t^2(\dot{B}_{p,1}^{\frac{d}{p}-1})} \lesssim \mathcal{X}_p^2(t).$$

For the term  $\tilde{K}_2(a)\theta \operatorname{div} v$ , we use the decomposition

$$\tilde{K}_2(a)\theta \operatorname{div} v = \tilde{K}_2(a)\theta^\ell \operatorname{div} v + \tilde{K}_2(a)\theta^h \operatorname{div} v.$$

Now, we have

$$(4.58) \quad \begin{aligned} \|\tilde{K}_2(a)\theta^\ell \operatorname{div} v\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h &\lesssim 2^{-j_0} \|\tilde{K}_2(a)\theta^\ell \operatorname{div} v\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}-1})}^h \\ &\lesssim (1 + \|a\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}})}) \|\theta^\ell\|_{L_t^2(\dot{B}_{p,1}^{\frac{d}{p}})} \|\operatorname{div} v\|_{L_t^2(\dot{B}_{p,1}^{\frac{d}{p}-1})} \lesssim \mathcal{X}_p^2(t) \end{aligned}$$

and

$$\begin{aligned} \|\tilde{K}_2(a)\theta^h \operatorname{div} v\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h &\lesssim (1 + \|a\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}})}) \|\operatorname{div} v\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-2})} \|\theta^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}})} \\ &\lesssim (\|v\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})}^\ell + \|v\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})}^h) \|\theta^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{d}{p}})} \lesssim \mathcal{X}_p^2(t). \end{aligned}$$

Therefore, putting together all the above estimates, we conclude that

$$(4.59) \quad \sum_{j \geq j_0-1} \sup_{0 \leq t \leq 2} \left( \langle t \rangle^\alpha \int_0^t e^{-c_0(t-\tau)} 2^{j(\frac{d}{p}-1)} S_j(\tau) d\tau \right) \lesssim \mathcal{X}_p^2(2).$$

To bound the supremum for  $2 \leq t \leq T$  in the last term of (4.48), one can split the integral on  $[0, t]$  into integrals on  $[0, 1]$  and  $[1, t]$ . The  $[0, 1]$  part can be handled exactly as the supremum on  $[0, 2]$  treated before. For the  $[1, t]$  part of the integral, we use the fact that

$$(4.60) \quad \sum_{j \geq j_0-1} \sup_{2 \leq t \leq T} \left( \langle t \rangle^\alpha \int_1^t e^{c_0(\tau-t)} 2^{j(\frac{d}{p}-1)} S_j(\tau) d\tau \right) \lesssim \sum_{j \geq j_0-1} 2^{j(\frac{d}{p}-1)} \sup_{1 \leq t \leq T} t^\alpha S_j(t).$$

In order to bound the term corresponding to  $S_j^1$ , we decompose  $a$  and  $v$  into low and high frequencies. Now, we obviously have

$$\begin{aligned} \|t^\alpha a v^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})} &\lesssim \|a\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}})} \|t^\alpha v\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})}^h \lesssim \mathcal{X}_p(t) \mathcal{D}_p(t) \\ \|t^\alpha a^h v\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})} &\lesssim \|t^\alpha a\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}})}^h \|v\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})} \lesssim \mathcal{X}_p(t) \mathcal{D}_p(t), \end{aligned}$$

and, using (4.54),

$$\begin{aligned}
\|t^\alpha a^\ell v^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})}^h &\lesssim \|t^\alpha a^\ell v^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}+1})}^h \\
&\lesssim \|t^\alpha a^\ell v^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \\
&\lesssim \|t^{\frac{1}{2}(s_1+\frac{d}{2}-\varepsilon)} a\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \|t^{\frac{1}{2}(s_1+\frac{d}{2}+\frac{1}{2}-\varepsilon)} v\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell \\
&\quad + \|t^{\frac{1}{2}(s_1+\frac{d}{2}-\varepsilon)} v\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \|t^{\frac{1}{2}(s_1+\frac{d}{2}+\frac{1}{2}-\varepsilon)} a\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell.
\end{aligned}$$

In order to bound the right-hand side, we notice that, from the definition of tilde norms and of  $\mathcal{D}_p$ , one can get for  $k = 0, 1, 2$ ,

$$\begin{aligned}
\|t^{\frac{s_1}{2}+\frac{d}{4}+\frac{k}{2}-\frac{1}{2}-\frac{\varepsilon}{2}}(D^k a^\ell, D^k v^\ell, D^k \theta^\ell)\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} &\lesssim \|t^{\frac{s_1}{2}+\frac{d}{4}+\frac{k}{2}-\frac{1}{2}-\frac{\varepsilon}{2}}(a, v, \theta)\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1+k-\varepsilon})}^\ell \\
(4.61) \qquad \qquad \qquad &\leq \mathcal{D}_p(T).
\end{aligned}$$

So finally, we have

$$\sum_{j \geq j_0-1} 2^{j(\frac{d}{p}-1)} \sup_{1 \leq t \leq T} t^\alpha S_j^1(t) \lesssim \mathcal{D}_p^2(T).$$

The terms corresponding to  $v \cdot \nabla v^h$ ,  $v^h \cdot \nabla v^\ell$ ,  $K_1(a) \nabla a^h$  and  $I(a) \Delta v^h$  may be bounded as  $av^h$  or  $a^h v$ . As regards  $v^\ell \cdot \nabla v^\ell$ , one can write using again (4.54) and (4.61),

$$\begin{aligned}
\|t^\alpha v^\ell \cdot \nabla v^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})}^h &\lesssim \|t^\alpha v^\ell \cdot \nabla v^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^h \\
&\lesssim \|t^{\frac{1}{2}(s_1+\frac{d}{2}-\varepsilon)} v^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \|t^{\frac{1}{2}(s_1+\frac{d}{2}+\frac{1}{2}-\varepsilon)} \nabla v^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \\
&\lesssim \mathcal{D}_p^2(T)
\end{aligned}$$

and similarly, thanks to (4.27),

$$\begin{aligned}
\|t^\alpha K_1(a) \nabla a^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})} &\lesssim \|t^{\frac{1}{2}(s_1+\frac{d}{2}-\varepsilon)} a\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}})}^\ell \|t^{\frac{1}{2}(s_1+\frac{d}{2}+\frac{1}{2}-\varepsilon)} \nabla a^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \\
\|t^\alpha I(a) \Delta v^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})} &\lesssim \|t^{\frac{1}{2}(s_1+\frac{d}{2}-\varepsilon)} a\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}})}^\ell \|t^{\frac{1}{2}(s_1+\frac{d}{2}+\frac{1}{2}-\varepsilon)} \Delta v^\ell\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})}^\ell.
\end{aligned}$$

The terms in  $S_j^4$ ,  $S_j^5$  and  $S_j^6$  may be treated along the same lines. So let us next focus on the terms of  $S_j^2$  and  $S_j^3$  corresponding to (4.50). Then Proposition 3.3 and Inequality (4.54) ensure that

$$\begin{aligned}
\|t^\alpha K_2(a) \nabla \theta^\ell\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})}^h &\lesssim \|t^{\frac{s_1}{2}+\frac{d}{4}-\frac{\varepsilon}{2}} K_2(a)\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}})}^\ell \|t^{\frac{s_1}{2}+\frac{d}{4}+\frac{1}{2}-\frac{\varepsilon}{2}} \nabla \theta^\ell\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}})}^\ell \\
&\lesssim \left( \|t^{\frac{s_1}{2}+\frac{d}{4}-\frac{\varepsilon}{2}} a\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \|t^{\frac{s_1}{2}+\frac{d}{4}-\frac{\varepsilon}{2}} a\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}})}^h \right) \|t^{\frac{s_1}{2}+\frac{d}{4}+\frac{1}{2}-\frac{\varepsilon}{2}} \nabla \theta\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell.
\end{aligned}$$

Consequently, we arrive at

$$(4.62) \qquad \|t^\alpha K_2(a) \nabla \theta^\ell\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})} \lesssim \mathcal{D}_p^2(T).$$

Additionally, it follows from Proposition 3.5 that

$$(4.63) \qquad \|t^\alpha K_2(a) \nabla \theta^h\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})} \lesssim \|a\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}})}^\ell \|t^\alpha \nabla \theta\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})}^h \lesssim \mathcal{X}_p(T) \mathcal{D}_p(T).$$

To bound  $t^\alpha \theta \nabla K_3(a)$ , we use that  $t^\alpha \theta \nabla K_3(a) = t^\alpha \theta^\ell \nabla K_3(a) + t^\alpha \theta^h \nabla K_3(a)$ . The second term can be estimated as in (4.63). For the first term, we write  $K_3(a) = K'_3(0)a + \widehat{K}_3(a)a$  for some smooth function  $\widehat{K}_3$  vanishing at 0. Now, we have thanks to (4.61),

$$\begin{aligned} \|t^\alpha \theta^\ell \nabla a^\ell\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})}^h &\lesssim \|t^{\frac{s_1}{2}+\frac{d}{4}-\frac{\varepsilon}{2}} \theta^\ell\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|t^{\frac{s_1}{2}+\frac{d}{4}+\frac{1}{2}-\frac{\varepsilon}{2}} \nabla a^\ell\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \lesssim \mathcal{D}_p^2(T), \\ \|t^\alpha \theta^\ell \nabla a^h\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})}^h &\lesssim \|\theta^\ell\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} \|t^\alpha \nabla a^h\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})} \lesssim \mathcal{X}_p(T) \mathcal{D}_p(T), \end{aligned}$$

and, using Proposition 3.5, the the fact that  $\widehat{K}_3(0) = 0$  and (4.27),

$$\begin{aligned} \|t^\alpha \theta^\ell \nabla(\widehat{K}_3(a)a)\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})}^h &\lesssim \|t^{\frac{s_1}{2}+\frac{d}{4}-\frac{\varepsilon}{2}} \theta^\ell\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|t^{\frac{s_1}{2}+\frac{d}{4}-\frac{\varepsilon}{2}} a\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}})} \|t^{\frac{1}{2}} a\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}})} \\ &\lesssim \mathcal{D}_p^3(T), \end{aligned}$$

because  $\frac{s_1}{2} + \frac{d}{4} - \frac{\varepsilon}{2} > \frac{1}{2}$  for sufficiently small  $\varepsilon$ . In summary, we get

$$(4.64) \quad \|t^\alpha \theta \nabla K_3(a)\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})}^h \lesssim \mathcal{D}_p^2(T) + \mathcal{D}_p^3(T) + \mathcal{X}_p(T) \mathcal{D}_p(T).$$

Likewise, Proposition 3.3 implies that

$$(4.65) \quad \|t^\alpha v \cdot \nabla \theta^h\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h \lesssim \|v\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})} \|t^\alpha \nabla \theta^h\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})} \lesssim \mathcal{X}_p(T) \mathcal{D}_p(T),$$

and it follows from (4.54), (4.61) and (4.27) that

$$\begin{aligned} \|t^\alpha v \cdot \nabla \theta^\ell\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h &\lesssim \|t^{\frac{s_1}{2}+\frac{d}{4}-\frac{\varepsilon}{2}} v\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}})} \|t^{\frac{s_1}{2}+\frac{d}{4}+\frac{1}{2}-\frac{\varepsilon}{2}} \nabla \theta^\ell\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \\ &\lesssim \mathcal{D}_p^2(T). \end{aligned}$$

Next, we have

$$(4.66) \quad \|t^\alpha I(a) \Delta \theta^h\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h \lesssim \|a\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}})} \|t^\alpha \theta^h\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}})}^h \lesssim \mathcal{X}_p(T) \mathcal{D}_p(T)$$

and, using (4.27),

$$(4.67) \quad \|t^\alpha I(a) \Delta \theta^\ell\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h \lesssim \|t^{\frac{s_1}{2}+\frac{d}{4}+\frac{1}{2}-\frac{\varepsilon}{2}} \Delta \theta^\ell\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} \|t^{\frac{s_1}{2}+\frac{d}{4}-\frac{\varepsilon}{2}} a\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}})}^h \lesssim \mathcal{D}_p^2(T).$$

To bound the term containing  $\frac{Q(\nabla v, \nabla v)}{1+a}$ , we just write that, owing to Propositions 3.3, 3.5, and to Condition (4.1),

$$\left\| t^\alpha \frac{Q(\nabla v, \nabla v)}{1+a} \right\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h \lesssim \left( \|t^{\frac{s_1}{2}+\frac{d}{4}+\frac{1}{4}-\frac{\varepsilon}{2}} \nabla v\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}})}^\ell + \|t^{\frac{s_1}{2}+\frac{d}{4}+\frac{1}{4}-\frac{\varepsilon}{2}} \nabla v\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}})}^h \right)^2.$$

Hence, from (4.1), (4.61) and the definition of  $\mathcal{D}_p$ , we infer that

$$(4.68) \quad \left\| t^\alpha \frac{Q(\nabla v, \nabla v)}{1+a} \right\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h \lesssim \mathcal{D}_p^2(T).$$

For the next term in (4.50), we start with the decomposition

$$\tilde{K}_1(a) \operatorname{div} v \triangleq \tilde{K}_1(a) \operatorname{div} v^h + \tilde{K}_1(a) \operatorname{div} v^\ell.$$

Now, by virtue of (4.54), Propositions 3.3 and 3.5,

$$(4.69) \quad \|t^\alpha \tilde{K}_1(a) \operatorname{div} v^h\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h \lesssim \|a\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}})} \|t^\alpha v\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})}^h \lesssim \mathcal{X}_p(T) \mathcal{D}_p(T)$$

and, thanks to (4.27), (4.54) and (4.61),

$$\|t^\alpha \tilde{K}_1(a) \operatorname{div} v^\ell\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h \lesssim \|t^{\frac{s_1}{2} + \frac{d}{4} + \frac{1}{2} - \frac{\varepsilon}{2}} \operatorname{div} v^\ell\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|t^{\frac{s_1}{2} + \frac{d}{4} - \frac{\varepsilon}{2}} a\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}})} \lesssim \mathcal{D}_p^2(T).$$

For  $\tilde{K}_2(a) \theta \operatorname{div} v$ , as the regularity of  $\theta^h$  is lower than that of  $v^h$ , we use the decomposition  $\tilde{K}_2(a) \theta \operatorname{div} v = \tilde{K}_2(a) \theta^\ell \operatorname{div} v + \tilde{K}_2(a) \theta^h \operatorname{div} v$  again. Remembering (4.1), it follows from Propositions 3.3 and 3.5 and Inequality (4.61) that

$$\begin{aligned} \|t^\alpha \tilde{K}_2(a) \theta^h \operatorname{div} v\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h &\lesssim \|t^\alpha \theta^h\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}})} \|\operatorname{div} v\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-2})} \\ &\lesssim \mathcal{D}_p(T) \mathcal{X}_p(T) \\ \|t^\alpha \tilde{K}_2(a) \theta^\ell \operatorname{div} v\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h &\lesssim \|t^{\frac{s_1}{2} + \frac{d}{4} - \frac{\varepsilon}{2}} \theta\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell \|t^{\frac{s_1}{2} + \frac{d}{4} + \frac{1}{2} - \frac{\varepsilon}{2}} \operatorname{div} v\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}})} \\ &\lesssim \mathcal{D}_p^2(T). \end{aligned}$$

Putting all above inequalities together, the r.h.s. of (4.60) can be estimated as follows:

$$\sum_{j \geq j_0-1} 2^{j(\frac{d}{p}-1)} \sup_{1 \leq t \leq T} t^\alpha S_j(t) \lesssim \mathcal{X}_p^2(T) + \mathcal{D}_p^3(T) + \mathcal{D}_p^2(T) + \mathcal{D}_p(T) \mathcal{X}_p(T).$$

Consequently, keeping in mind that  $\mathcal{D}_p$  is small, we obtain

$$\begin{aligned} (4.70) \quad \|\langle t \rangle^\alpha (\nabla a, v)\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})}^h &+ \|\langle t \rangle^\alpha \theta\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h \\ &\lesssim \|(\nabla a_0, v_0)\|_{\dot{B}_{p,1}^{\frac{d}{p}-1}}^h + \|\theta_0\|_{\dot{B}_{p,1}^{\frac{d}{p}-2}}^h + \mathcal{X}_p^2(T) + \mathcal{D}_p^2(T). \end{aligned}$$

*Step 3: Decay estimates with gain of regularity for the high frequencies of  $v$  and  $\theta$ .* We here want to prove that the parabolic smoothing effect provided by the last two equations of (2.1) allows to get gain of regularity and decay altogether for  $v$  and  $\theta$ . Let us focus on the equation for  $\theta$  (handling  $v$  being similar). Recall that

$$\partial_t \theta - \beta \Delta \theta = -\gamma \operatorname{div} v + k.$$

Hence

$$(4.71) \quad \partial_t(t^\alpha \Delta \theta) - \beta \Delta(t^\alpha \Delta \theta) = \alpha \beta t^{\alpha-1} \Delta \theta + \beta t^\alpha \Delta(k - \gamma \operatorname{div} v), \quad t^\alpha \Delta \theta|_{t=0} = 0.$$

We thus deduce from Proposition 3.1 that

$$(4.72) \quad \|\tau^\alpha \nabla^2 \theta\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h \lesssim \|\tau^{\alpha-1} \Delta \theta\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-4})}^h + \|\tau^\alpha \Delta(k - \gamma \operatorname{div} v)\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-4})}^h,$$

hence,

$$(4.73) \quad \|\tau^\alpha \nabla \theta\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})}^h \lesssim \|\tau^{\alpha-1} \theta\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h + \|\tau^\alpha v\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})}^h + \|\tau^\alpha k\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h.$$

Because  $\alpha \geq 1$ , we have

$$(4.74) \quad \|\tau^{\alpha-1} \theta\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h \lesssim \|\langle \tau \rangle^\alpha \theta\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h \quad \text{and} \quad \|\tau^\alpha v\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})}^h \lesssim \|\langle \tau \rangle^\alpha v\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})}^h.$$

Bounding  $\|\tau^\alpha k\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-2})}^h$  as in Step 2, one can thus conclude that

$$(4.75) \quad \|\tau^\alpha \nabla \theta\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})}^h \lesssim \|\langle \tau \rangle^\alpha v\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1})}^h + \mathcal{D}_p^2(t) + \mathcal{X}_p^2(t).$$



Arguing similarly with the second equation of (2.1), we get

$$(4.76) \quad \|\tau^\alpha \nabla v\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}})}^h \lesssim \|\langle \tau \rangle^\alpha(a, \theta)\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{d}{p}})}^h + \mathcal{D}_p^2(t) + \mathcal{X}_p^2(t).$$

Finally, bounding the first terms on the right-side of (4.75)-(4.76) according to (4.70), and adding up (4.75)-(4.76) to (4.40) and (4.70) yields for all  $T \geq 0$ ,

$$\mathcal{D}_p(T) \lesssim \mathcal{D}_{p,0} + \|(a_0, v_0, \theta_0)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^\ell + \|(\nabla a_0, v_0)\|_{\dot{B}_{p,1}^{\frac{d}{p}-1}}^h + \|\theta_0\|_{\dot{B}_{p,1}^{\frac{d}{p}-2}}^h + \mathcal{X}_p^2(T) + \mathcal{D}_p^2(T).$$

As Theorem 1.1 ensures that  $\mathcal{X}_p \lesssim \mathcal{X}_{p,0}$  and as

$$\|(a_0, u_0, \theta_0)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^\ell \lesssim \|(a_0, u_0, \theta_0)\|_{\dot{B}_{2,\infty}^{-s_1}}^\ell,$$

one can conclude that (2.4) is fulfilled for all time if  $\mathcal{D}_{p,0}$  and  $\mathcal{X}_{p,0}$  are small enough. This completes the proof of Theorem 2.1.  $\square$

Corollary 2.1 easily follows from Theorem 2.1 : let us just show the inequality for  $\theta$  as an example. Since the embedding  $\dot{B}_{2,1}^s \hookrightarrow \dot{B}_{p,1}^{s-d(1/2-1/p)} \hookrightarrow \dot{B}_{p,1}^s$  holds for the low frequencies whenever  $p \geq 2$ , we have for all  $-s_1 < s \leq \frac{d}{p} - 2$ ,

$$\sup_{t \in [0, T]} \langle t \rangle^{\frac{s_1+s}{2}} \|\Lambda^s \theta\|_{\dot{B}_{p,1}^0}^\ell \lesssim \|\langle t \rangle^{\frac{s_1+s}{2}} \theta\|_{L_T^\infty(\dot{B}_{2,1}^s)}^\ell + \|\langle t \rangle^{\frac{s_1+s}{2}} \theta\|_{L_T^\infty(\dot{B}_{p,1}^s)}^h.$$

Hence, using (2.4) yields

$$\|\langle t \rangle^{\frac{s_1+s}{2}} \theta\|_{L_T^\infty(\dot{B}_{2,1}^s)}^\ell \lesssim (\mathcal{D}_{p,0} + \|(\nabla a_0, v_0)\|_{\dot{B}_{p,1}^{\frac{d}{p}-1}}^h + \|\theta_0\|_{\dot{B}_{p,1}^{\frac{d}{p}-2}}^h).$$

Now, the fact that  $\alpha \geq \frac{s_1+s}{2}$  for all  $s \leq \frac{d}{p} - 2$  allows to write that

$$\|\langle t \rangle^{\frac{s_1+s}{2}} \theta\|_{L_T^\infty(\dot{B}_{p,1}^s)}^h \lesssim (\mathcal{D}_{p,0} + \|(\nabla a_0, v_0)\|_{\dot{B}_{p,1}^{\frac{d}{p}-1}}^h + \|\theta_0\|_{\dot{B}_{p,1}^{\frac{d}{p}-2}}^h),$$

which completes the proof of Corollary 2.1.  $\square$

We end this section with some heuristics concerning the optimality of the regularity and decay exponents in the definition of  $\mathcal{D}_p$ . Let us first explain why the regularity exponent  $s$  in  $\mathcal{D}_{p,1}$  has to satisfy  $s \leq \frac{d}{2} + 1$ . The general fact (based on Inequality (4.23)) that we used repeatedly is that the time decay exponent  $\delta$  for  $\|(f, g, k)\|_{\dot{B}_{2,\infty}^{-s_1}}$  must satisfy  $\delta \geq \frac{s_1+s}{2}$ .

Now, if we look at the term  $a^\ell \operatorname{div} u^\ell$ , then a necessary condition for having

$$\|a^\ell \operatorname{div} u^\ell\|_{\dot{B}_{2,\infty}^{-s_1}}^\ell \lesssim \|a^\ell\|_{\dot{B}_{2,\infty}^{\sigma_1}} \|\operatorname{div} u^\ell\|_{\dot{B}_{2,\infty}^{\sigma_2}}$$

is that  $\sigma_1 + \sigma_2 \leq \frac{d}{2} - s_1$ . As the decay exponent of the right-hand side is  $s_1 + \frac{\sigma_1 + \sigma_2 + 1}{2}$ , we deduce that  $\delta \leq \frac{d}{4} + \frac{s_1}{2} + \frac{1}{2}$ . Hence we must have  $s \leq \frac{d}{2} + 1$ .

To see that the decay rate in  $\mathcal{D}_{p,2}$  cannot be more than  $s_1 + \frac{d}{2} + \frac{1}{2}$ , one can observe that, owing to  $s \leq \frac{d}{2} + 1$ , the term  $a^\ell \nabla a^\ell$  (which at most has the same regularity as  $\nabla a^\ell$ ) cannot be estimated in a space with higher regularity than  $\dot{B}_{2,1}^{\frac{d}{2}}$ . As the corresponding estimate reads

$$\|a^\ell \nabla a^\ell\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \lesssim \|a^\ell\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla a^\ell\|_{\dot{B}_{2,1}^{\frac{d}{2}}},$$

the definition of  $\mathcal{D}_{p,1}$  ensures that the right-hand side has decay exponent  $s_1 + \frac{d}{2} + \frac{1}{2}$ . A similar argument shows that the decay rate in the definition of  $\mathcal{D}_{p,3}$  is optimal.

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UNIVERSITÉ PARIS-EST, LAMA (UMR 8050), UPEMLV, UPEC, CNRS, 61 AVENUE DU GÉNÉRAL DE GAULLE, 94010 CRÉTEIL CEDEX 10

*E-mail address:* danchin@univ-paris12.fr

DEPARTMENT OF MATHEMATICS, NANJING UNIVERSITY OF AERONAUTICS AND ASTRONAUTICS, NANJING 211106, P.R.CHINA,

*E-mail address:* jiangxu\_79math@yahoo.com